Common Idiosyncratic Quantile Risk*

Jozef Baruník[a,b], and Matěj Nevrla[a,b†]

[a] Institute of Economic Studies, Charles University,
Opletalova 26, 110 00, Prague, Czech Republic

[b] The Czech Academy of Sciences, Institute of Information Theory and Automation
Pod Vodarenskou Vezi 4, 182 00, Prague, Czech Republic

February 19, 2024

Abstract

We identify a new type of risk that is characterised by commonalities in the quantiles of the cross-sectional distribution of asset returns. Our newly proposed quantile risk factor is associated with a quantile-specific risk premium and provides new insights into how upside and downside risks are priced by investors. In contrast to the previous literature, we recover the common structure in cross-sectional quantiles without making confounding assumptions or aggregating potentially non-linear information. We discuss how the new quantile-based risk factor differs from popular volatility and downside risk factors, and we identify where the quantile-dependent risks deserve greater compensation. Quantile factors also have predictive power for aggregate market returns.

Keywords: Cross-section of asset returns, factor structure of asset returns, idiosyncratic risk, quantiles, asymmetric risk
JEL: C21; C58; G12

*We appreciate helpful comments from participants at various seminars, workshops and conferences. Support from the Czech Science Foundation under the 19-28231X (EXPRO) project is gratefully acknowledged.
†Corresponding author, Tel. +420(734)593183, Email address: matej.nevrla@fsv.cuni.cz

1 Introduction

The question of how relevant the information contained in different parts of the return distribution is to an investor has received considerable attention in the recent empirical asset pricing literature (Ang et al., 2006; Van Oordt and Zhou, 2016; Chabi-Yo et al., 2018; Lu and Murray, 2019), with number of studies focusing on the tails or extremes in the cross-section of returns (Kelly and Jiang, 2014; Chabi-Yo et al., 2022). These studies typically rely on assumptions about moment conditions as well as the existence of a model that generates returns. In contrast to the literature, our aim is to use conditional quantiles of observed returns to capture set of nonlinear factors that provide finer characterization of risk. In particular, we want to explore the common, possibly non-linear movements in the panel of the firm's idiosyncratic quantiles. In doing so, we remain agnostic about the data generating process. We believe that such structures provide richer information for investors than the information that can be obtained by making assumptions about the moments. In particular, we will identify where quantile-dependent risk exposures deserve greater compensation. Both volatility and downside risk measures hide such details while aggregating information about risk.

The information captured by quantile-dependent factors can be related to the behaviour of investors with quantile preferences (de Castro and Galvao, 2019). Quantiles contain rich information because they capture heterogeneity in risk and allow the separation of risk aversion and elasticity of intertemporal substitution. Our main interest is to show that there are strong common factors across quantiles of the cross-sectional distribution of asset returns that are more informative about investors' compensation requirements. We argue that such risk is distinct from other types of risk associated with the distribution of returns, such as downside risk or volatility risk. The quantile-dependent risk premia associated with such factors are then used to generalise notion of upside risk and downside risk.

Just as quantile regression extends classical linear regression, our quantile factor model of asset returns extends the approximate factor models used in the empirical asset pricing literature. In the spirit of the popular Principal Component Analysis, which recovers the conditional mean, we work with more general quantile factor models (QFMs). These are flexible enough to capture quantile-dependent objects that cannot be captured by standard tools. Unlike standard principal component analysis, quantile factor models are able to capture hidden factors that shift distributional properties such as moments or quantiles. Moreover, these factors can vary across the distribution of each unit in the panel, allowing the factors to be properly inferred when the idiosyncratic error distributions have heavy tails. Importantly, such factors differ from the usual mean and volatility factors when we abandon the traditional location and scale shift model structure and allow for more general, possibly unknown, data generating processes. In effect, quantile-dependent risk is treated as constant in factor models based on such assumptions. Downside risk models then aggregate the quantiles, usually under some distributional assumption.

Our main contribution is to investigate the pricing implications of common non-linear factors that are quantile specific for the predictability of aggregate market returns and the cross-section of stock returns. We are interested in factors that identify the risk premium associated with different quantiles of the return distribution in terms of both downside (or tail) risk and upside potential. Our approach will identify new information about risk beyond the usual moments associated with tail risks. To this end, we use the quantile factor model of Chen et al. (2021) and investigate the pricing implications of quantile-dependent factors while controlling for various linear factors and exposures to them. Our objective is also motivated by the increasing evidence of non-linearities in equity markets.[1] We aim to show that the common quantile risk present in the stock return data carries different information from the common volatility and downside risks. Our quantile dependent factors also carry strong information for both the cross-section of asset returns and the time series predictability of the equity premium.

We begin by identifying common factor structures in the idiosyncratic quantiles of stocks in the Center for Research in Security Prices (CRSP) over a sample spanning 1960 to 2018. We discuss the relationship with volatility and downside risk factors and show that quantile factors have predictive power for aggregate market returns. Predictive regressions show that a one standard deviation increase in quantile risk predicts a statistically significant increase in annualised excess market returns of up to 7.05% in the case of the left tail. These results hold out-of-sample, are stronger for the left tail, and are robust to controlling for a wide range of popular predictors studied by Welch and Goyal (2007), as well as tail risk (Kelly and Jiang, 2014), common volatility risk (Herskovic et al., 2016), and variance risk premium (Bollerslev et al., 2009). We also document the predictive power of the upper tail factor with a smaller effect of up to 3.50% increase in annualised returns, hence the effect is asymmetric. Moreover, the predictive power of the upper tail factors disappears when looking at the out-of-sample performance.

We also find that idiosyncratic quantile risk has significant predictive power for the cross-section of average returns. We show that stocks with high loadings of past quantile risk in the left tail earn up to an annual six-factor alpha of 8.57% higher than stocks with low tail risk

[1] E.g., Amengual and Sentana (2020) report a non-linear dependence structure in short-term reversals and momentum. Ma et al. (2021) show that many firm-level characteristics have a complex relationship with returns in terms of quantiles.

loadings for 0.2 quantiles. This risk premium is not subsumed by other commonly priced factors such as common volatility, tail and downside risk, and other popular risk factors. Investors thus have a strong aversion to tail risk with respect to the common movements in idiosyncratic returns. On the other hand, the absence of the risk premium associated with the factors for the upper quantiles suggests that investors are not upside potential seekers. Both results are consistent with the literature on the impact of asymmetric dependencies on asset prices.

Our work is related to several strands of the literature. The first relates to the factor-based asset pricing models that are very popular in the empirical pricing literature (Ross, 1976; Fama and French, 1993; Kelly et al., 2019). In sharp contrast to this literature, our approach remains agnostic about the nature of the true data generating process and uses the conditional quantiles of observed returns without imposing moment conditions.

The second strand to which we contribute is the study of idiosyncratic risk that co-moves across assets, thus exploring common trends that are not captured by first moment factors. The bulk of this research is motivated by the introduction of the idiosyncratic volatility puzzle proposed by Ang et al. (2006a). Unfortunately, all existing explanations of the anomaly are based on lottery preferences, market frictions or other factors[2] only for 29-54% of the puzzle using individual stocks Hou and Loh (2016).

The third line of thought that we take into account deals with asymmetric properties of systematic risk and how they are incorporated into asset prices. Interest in this type of model was reignited by Ang et al. (2006) and their introduction of downside beta, which captures the covariance between asset and market returns conditional on the market being below some threshold. Bollerslev et al. (2021) further decompose traditional market beta into semibetas, which are characterised by the signed covariation between market and asset returns. They show that only the semibetas associated with negative market and asset returns predict significantly higher future returns. More recently, Bollerslev et al. (2022) argue that betas are granular and associated with a risk premium that depends on the relevant part of the return distributions.

From a theoretical point of view, there are many justifications for the departure from classical common factor pricing theory to the asymmetric forms of the utility function. Probably the most relevant for our work is the dynamic quantile decision maker of de Castro and Galvao (2019), who decides based on quantile dependent preferences. Barro (2006), building on

[2]For a comprehensive list of references belonging to each of these categories, see Hou and Loh (2016). The only exception to this observation is the lottery-based explanation using the highest realised return from the previous month, proposed by Bali et al. (2011) and confirmed in European markets by Annaert et al. (2013). However, Hou and Loh (2016) argue that this explanation is not valid as it is an almost perfect collinear range-based measure of idiosyncratic volatility.

Rietz (1988), introduced the rare disaster model and showed that tail events may have significant ability to explain various asset pricing puzzles, such as the equity premium puzzle. The other popular model that considers asymmetric features of risk is the generalised disappointment aversion model of Routledge and Zin (2010), which inherently assumes that investors are downside averse. Based on these preferences, Farago and Tédongap (2018) introduced an intertemporal equilibrium asset pricing model and showed that the disappointment-related factors should be priced in the cross-section. Moreover, they prove that their model performs well empirically by jointly pricing different asset classes with significant prices for the risk associated with the disappointment factors.

There are also attempts to combine the two or three of these research agendas. Herskovic et al. (2016) introduced a risk factor based on the common volatility of firm-level idiosyncratic returns, and showed its pricing capabilities for the cross section of different asset classes. For example, Kelly and Jiang (2014); Allen et al. (2012); Jondeau et al. (2019) explore the risks associated with skewness, tails and extremes. Giglio et al. (2016) estimate quantile-specific latent factors using systemic risk and financial market distress variables to predict macroeconomic activity. Much of the research investigating common tail risk and its implications for asset pricing relies on options data. They argue that the tail factor identifies additional information beyond the volatility factor. Andersen et al. (2020) show strong predictive power for future equity risk premia in US and European equity index derivatives. Bollerslev and Todorov (2011) combine high-frequency and options data and use a nonparametric approach to conclude that a large part of the equity and variance risk premia is related to jump tail risk.

The rest of the paper is structured as follows. Section 2 proposes the quantile factor model for asset returns, discusses the methodology of estimating the quantile-specific factors and the data we use, and provides the link to the volatility factors. Section 3 presents the results on the time series predictability of the aggregate market return using the common idiosyncratic quantile factors. Section 4 examines the cross-sectional asset pricing implications of the proposed factors. Section 5 concludes.

2 Common Idiosyncratic Quantile Factors

Researchers usually assume that time variation in equity returns can be captured by relatively small number of common factors with following structure[3]

$$r_{i,t} = \alpha_i + \beta_i^\top f_t + \epsilon_{i,t} \tag{1}$$

where $r_{i,t}$ is excess return of an asset $i = 1, \ldots, N$ at time $t = 1, \ldots, T$, f_t is a $k \times 1$ vector of common factors and β_i is a $k \times 1$ vector of the asset's i exposures to the common factors. Such time-series regressions as the one in (1) yielding high R^2 are used to identify factors serving as good proxies for aggregate risks present in the economy. Exposures to the relevant factors captured by β_i coefficients should be compensated in the equilibrium and explain the risk premium of the assets

$$\mathbb{E}_t[r_{i,t+1}] = \beta_i^\top \lambda_t \tag{2}$$

where the λ_t is a $k \times 1$ vector of prices of risk associated with factor exposures. Importantly, while the arbitrage pricing theory (APT) of Ross (1976) suggests that any common return factors f_t are valid candidate asset pricing factors, the idiosyncratic return residuals $\epsilon_{i,t}$ are assumed not to be priced. This implication is due to many simplifying assumptions, such that an average investor can perfectly diversify her portfolio or that the linear model (1) is correctly specified.

In these models, only common return factors are valid candidate pricing factors, and sensitivities to those factors determine the risk premium associated with an asset (Ross, 1976). This strand of literature yields highly successful and popular results focusing on the parsimonious models (Fama and French, 1993), as well as exploration of statistically motivated latent factors.[4] Recently, Kelly et al. (2019) introduced instrumented principal component analysis, which enables to flexibly model the latent factors with time-varying loadings using the observable characteristics.[5] In addition, Ma et al. (2021) introduced

[3]Recently, Lettau and Pelger (2020) introduce Risk-Premium Principal Component Analysis that allows for systematic time-series factors incorporating information from the first and second moment.

[4]This approach dates back to Chamberlain and Rothschild (1983) and Connor and Korajczyk (1986). For a comprehensive overview of machine learning methods applied to asset pricing problems such as measuring expected returns, estimating factors, risk premia, or stochastic discount factor, model selection, and corresponding asymptotic theory, see Giglio et al. (2022).

[5]Other notable recent contributions to the factor literature are, e.g., Kozak et al. (2018) and Giglio et al. (2021). The recent availability of high-frequency return data also motivated the development of continuous-time factor models.Aït-Sahalia et al. (2020) proposed a generalization of the classical two-pass Fama-MacBeth regression from the classical discrete-time factor setting to a continuous-time factor model and enables uncovering complex dynamics such as jump risk and its role in the expected returns.

a semi-parametric quantile factor panel model that considers stock-specific characteristics, which may non-linearly affect stock returns in a time-varying manner. They find that many characteristics possess a non-linear effect on stock returns. In contrast to these authors, the approach used in our paper is more general since it allows not only loadings but also factors to be quantile-dependent. Moreover, our approach does not require the loadings to depend on observables and has direct relation of the approximate factor models that are ubiquitous in the finance literature.

While large literature have focused mainly on the diversification assumption, we aim to question linear nature of the factor model, and our focus is on exposure to parts of idiosyncratic return's distribution instead. Recently, Herskovic et al. (2016) documents strong comovement in idiosyncratic volatility that does not arise from omitted factors, and even after saturating the factor regression with up to ten principal components, residuals that are virtually uncorrelated display same co-movement seen in raw returns.

While the exposure to common movements in volatility seem to carry strong pricing implications, we ask if there exist additional structure insufficiently captured by volatilities especially in a non-linear and heavy tailed financial data. In other words, we ask if various parts of the return distributions may have pricing implications for the cross-section of stock returns.[6]

In parallel to simple factor structure in idiosyncratic volatility of a panel of returns recovered commonly by researchers (Ang et al., 2006b; Herskovic et al., 2016), we aim to recover genuine unobserved structure in idiosyncratic quantiles. These quantities will be more informative for investors in case of the heavy-tailed nonlinear data in which the second moment is not sufficient quantity for capturing risk. We will show the relation of quantile factors to volatility under some specific model assumptions, relate the proposed factor model to existing approaches recovering various factor structures from data and also provide a first look at the quantile factor structures in cross-section of the U.S. stocks. Importantly, we will show that our quantile dependent factors carry different information from the structure recovered using volatility or some popular downside risk measures that require certain moment conditions to be met.

[6]Ando and Bai (2020) document that the common factor structures explaining the upper and lower tails of the asset return distributions in global financial markets have become different since the subprime crisis.

2.1 Quantile Factor Model

To formalize the discussion, we assume the panel of returns of length T and width N after elimination of common mean factors from the time-series regression

$$r_{i,t} = \alpha_i + \beta_i^\top f_t + \epsilon_{i,t} \tag{3}$$

to have τ-dependent structure $f_t(\tau)$ in idiosyncratic errors that we coin common idiosyncratic quantile – CIQ(τ) – factors, $f_t(\tau)$

$$Q_{\epsilon_{i,t}}\Big[\tau|f_t(\tau)\Big] = \gamma_i^\top(\tau) f_t(\tau), \tag{4}$$

that implies

$$\epsilon_{i,t} = \gamma_i^\top(\tau) f_t(\tau) + u_{i,t}(\tau), \tag{5}$$

where $f_t(\tau)$ is an $r(\tau) \times 1$ vector of random common factors, and $\gamma_i(\tau)$ is $r(\tau) \times 1$ vector of non-random factor loadings with $r(\tau) \ll N$ and the quantile-dependent idiosyncratic error $u_{i,t}(\tau)$ satisfies the quantile restriction $P[u_{i,t}(\tau) < 0|f_t(\tau)] = \tau$ almost surely for all $\tau \in (0, 1)$.

To estimate the common factors that capture co-movement of quantile-specific features of distributions of the idiosyncratic parts of the stock returns, we use Quantile Factor Analysis (QFA) introduced by Chen et al. (2021). In contrast to the principal component analysis (PCA), QFA allows to capture hidden factors that may shift more general characteristics such as moments or quantiles of the distribution of returns other than mean. The methodology is also suitable for large panels and requires less strict assumptions about the data generating process as we will discuss in detail here.

The quantile-dependent factors and its loadings can be estimated as

$$\underset{(\gamma_1,\dots,\gamma_N,f_1,\dots,f_T)}{\operatorname{argmin}} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \rho_\tau\left(\epsilon_{it} - \gamma_i^\top f_t\right) \tag{6}$$

where $\rho_\tau(u) = (\tau - \mathbf{1}\{u \leq 0\})u$ is the check function while imposing the following normalizations $\frac{1}{T}\sum_{t=1}^{T} f_t f_t^\top = \mathbb{I}_r$, and $\frac{1}{N}\sum_{i=1}^{N} \gamma_i \gamma_i^\top$ is diagonal with non-increasing diagonal elements. A potential problem that may arise in small samples is the so-called quantile crossing, that is, the estimated quantiles are not guaranteed to be monotonic in τ. If this occurs, the approach due to Chernozhukov et al. (2010) can be employed to establish monotonicity of the estimated quantiles. In our empirical applications reported later, quantile crossing never arises.

As discussed in Chen et al. (2021), this estimator is related to the principal component analysis (PCA) estimator studied in Bai and Ng (2002) and Bai (2003) similarly as quantile regression is related to classical least-square regression. Unlike the PCA estimator of Bai (2003), the estimator does not yield an analytical closed form solution. To solve for the stationary points of the objective function, Chen et al. (2021) proposed a computational algorithm called iterative quantile regression. Moreover, they show that the estimator possess same convergence rate as the PCA estimators for approximate factor model. We follow their approach when estimating the quantile factors.[7]

It is important here to make relation to the recent literature that attempts to recover possibly non-linear commonalities and dependence structures in cross-section of returns. For example Pelger and Xiong (2022) allowed factors to be state-dependent, Chen et al. (2009) provided theory for nonlinear factors and Gorodnichenko and Ng (2017) estimated joint level and volatility factors simultaneously. Important strand of the literature is using copulas and documents nonlinear tail dependence, co-skewness, and co-kurtosis in cross-sectional dependence among monthly returns on individual U.S. stocks (Amengual and Sentana, 2020) or provides flexible copula factor model (Oh and Patton, 2017) .

Different from these studies, our model remains agnostic about the nature of the true data generating process, and use the conditional quantiles of the observed data to capture nonlinearities in factor models. In contrast to the literature, we also do not require the idiosyncratic errors to satisfy certain moment conditions. Hence our approach is more flexible as it estimates factors shifting relevant parts of the return distributions without restricting assumptions, relying on the properties of the density. The approach also departs from existing factor literature in not requiring the loadings to depend on observables and considers the factors to be quantile-dependent objects.

2.2 Relation to common factors in volatility

Quantiles of stock returns can be related to variety of quantities as well as distributional characteristics in specific cases. A specifically important quantity in finance that can relate to quantiles of the return distribution for a typically assumed location-scale model is volatility. As discussed by ample literature started by Ang et al. (2006b), there exists genuine factor structure in the idiosyncratic volatility of panel of asset returns. Applying PCA (or cross-sectional averages) to squared residuals, once mean factors have been removed from the returns (a procedure labeled PCA-SQ hereafter) will recover that structure. We will use this approach to study the relation to quantile specific factors on data, but before we do so, let's

[7] We employ the authors' Matlab codes provided on the Econometrica webpage.

discuss the relation theoretically.

It is important to note that the volatility structure will be recovered only if the data-generating process were to be known, and well characterized by the first two moments of the distribution. Yet in case of more general, or even unknown data generating processes that will not be well characterized by the first two moments, such approaches will fail to characterize the risks precisely, and quantile factor models will estimate more useful information.

To illustrate the discussion and provide the link between volatility and quantiles in such restrictive models, let's consider the data generating process to be a typical location-scale model with two unrelated factors in the first and second moments. Idiosyncratic returns $\epsilon_{i,t}$ of such model will be zero mean i.i.d. process independent of both factors with cumulative distribution function $F_{\epsilon_{i,t}}$. Further let $Q_{\epsilon_{i,t}}(\tau) = F^{-1}_{\epsilon_{i,t}}(\tau) = \inf\{s : F_{\epsilon_{i,t}}(s) \leq \tau\}$ be a quantile function of $\epsilon_{i,t}$ and assume the median is zero. Then the following model that is typical for finance

$$r_{i,t} = \beta_i f_{1,t} + (\sigma_{i,t}^\top f_{2,t})\epsilon_{i,t}, \tag{7}$$

where $\sigma_{i,t}$ is time-varying volatility of an ith stock and $\sigma_{i,t} f_{2,t} > 0$ can be assumed to generate returns. When $f_{1,t}$ and $f_{2,t}$ do not share common elements, then

$$Q_{r_{i,t}}\Big[\tau|f_t(\tau)\Big] = \beta_i f_{1,t} + \sigma_{i,t}^\top f_{2,t} Q_{\epsilon_{i,t}}(\tau) \tag{8}$$

for $\tau \neq 0.5$ and $Q_{r_{i,t}}\Big[\tau|f_t(\tau)\Big] = \beta_i f_{1,t}$ for $\tau = 0.5$. Note that here loadings on the factor are the only quantile-dependent objects and structure in the mean and volatility describes well the structure in quantiles. While this is already restrictive example that operates with the assumption on first two moments, even in such case standard PCA will not provide consistent estimates if the distribution of $\epsilon_{i,t}$ is heavy-tailed (Chen et al., 2021).

But what if the data follows more complicated models than the one implied by location-shift models? Consider adding asymmetric dependence such as

$$r_{i,t} = \beta_i f_{1,t} + f_{2,t}\epsilon_{i,t} + f_{3,t}\epsilon_{i,t}^3, \tag{9}$$

where $\epsilon_{i,t}$ is standard normal random variable with cumulative distribution function $\Phi(.)$. The quantiles of the returns will then follow

$$Q_{r_{i,t}}\Big[\tau|f_t(\tau)\Big] = \beta_i f_{1,t} + \Phi^{-1}(\tau)\big[f_{2,t} + f_{3,t}\Phi^{-1}(\tau)^2\big], \tag{10}$$

for $\tau \neq 0.5$ and we can clearly see that second factor in $f(\tau) = [f_{1,t}, f_{2,t} + f_{3,t}\Phi^{-1}(\tau)^2]^\top$ is quantile dependent.

The main benefit of the model proposed is that being agnostic about data generating process and moment conditions, we use conditional quantiles of the observed returns to capture nonlinearities in factor models. In case these factors are different from those obtained on first and second moments, they will also be more informative for investors. In the next section we estimate these quantities and compare them to volatility as well as other downside risk factors to find support that data show such a rich structures.

2.3 Common Idiosyncratic Quantile Factor and the US firms

To estimate the common idiosyncratic quantile – CIQ(τ) – factors, we use returns on stocks from the Center for Research in Securities Prices (CRSP) database sampled between January 1963 and December 2018. We include all stocks with codes 10 and 11 in estimating the CIQ(τ) factors. We adjust the returns for delisting as described in Bali et al. (2016). We follow the standard practice in the literature and exclude all "penny stocks" with prices less than one dollar to avoid biases related to these stocks.[8] We performed the analysis using all the stocks, and the results did not qualitatively change. When not stated otherwise, we use monthly data for both factor estimation and beta calculations.

In the process of the factor estimation, we proceed in a few steps. First, we use a moving window of 60 months of monthly sampled observations. We select the stocks that have all the observations in this window. For all these stocks, we run time-series regression to eliminate the influence of the common (linear) factors

$$\forall i : r_{i,t} = \alpha_i + \beta_i^\top f_t + e_{i,t}, \quad t = 1, \ldots, T \tag{11}$$

and save the residuals $e_{i,t}$. For the common factors f_t, which we eliminate from the stock returns, we resort to the three factors of Fama and French (1993).[9] Second, we use the residuals from the first step and, for every τ, estimate common idiosyncratic quantile factors, $f_t(\tau)$

$$\forall \tau : e_{i,t} = \gamma_i(\tau) f_t(\tau) + u_{i,t}(\tau) \tag{12}$$

where the quantile-dependent idiosyncratic error $u_{i,t}(\tau)$ satisfies the quantile restriction following the methodology discussed in the previous subsection. We use only the first – the

[8] See, e.g., Amihud (2002).

[9] As discussed in Herskovic et al. (2016), there is a little difference between the results obtained using factors of Fama and French (1993) and purely statistically motivated ones estimated using the PCA framework.

most informative – estimated factor for our purposes. In the overwhelming majority of the cases, the algorithms proposed in Chen et al. (2021) select exactly one factor to be the correct number of factors that explain the panels of idiosyncratic returns.

Since we are interested to see how the quantile dependent factors relate to volatility, we estimate an approximate factor model on squared residuals that captures the common volatility factor. More specifically, we use residuals obtained from the Equation 11, square them and estimate on them first principal component using PCA. Such factor denoted as PCA-SQ will fail to capture the full factor structure if the distribution of the idiosyncratic returns possess non-normal features (Chen et al., 2021).

While it is one of our main questions to study if quantile dependent risk is present in the markets, and is not subsumed by volatility and downside risk, we first look at the correlations between these risks. Consistent with common volatility factor literature, we also focus on the changes in the CIQ(τ), and we work with ΔCIQ(τ) factors.[10] Intuitively, we will look at how investors price the innovations of these risks rather then levels.

Table 1: *Correlations between CIQ(τ) and other factors.* The table reports correlations between CIQ(τ) factors and factors related to the asymmetric and variance risk. Data contain the period between January 1963 and December 2018.

variable / τ	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9
Panel A: *Levels of factors*											
PCA-SQ	-0.76	-0.73	-0.69	-0.56	-0.24	0.15	0.23	0.53	0.70	0.75	0.78
CIV	-0.45	-0.43	-0.39	-0.31	-0.06	-0.05	0.15	0.27	0.36	0.39	0.40
TR	0.13	0.12	0.12	0.07	0.01	-0.11	-0.11	-0.26	-0.27	-0.24	-0.23
VRP	-0.05	-0.04	-0.05	-0.02	0.04	-0.09	-0.03	0.07	0.08	0.08	0.09
VIX	-0.37	-0.34	-0.30	-0.20	0.12	0.11	0.20	0.36	0.40	0.39	0.39
Panel B: *Differences of factors*											
PCA-SQ	-0.53	-0.47	-0.43	-0.30	-0.09	0.21	0.22	0.37	0.53	0.59	0.65
CIV	-0.21	-0.20	-0.18	-0.15	-0.09	0.04	0.07	0.09	0.10	0.11	0.08
TR	0.04	0.03	0.03	-0.01	-0.08	-0.10	-0.15	-0.26	-0.29	-0.27	-0.25
VRP	0.12	0.11	0.11	0.06	0.08	-0.02	-0.04	-0.06	-0.08	-0.09	-0.10
VIX	0.24	0.25	0.27	0.27	0.26	0.04	0.07	0.10	0.02	-0.04	-0.11

Table 1 reports correlations between CIQ(τ) factors and factors related to the variance and asymmetric risk. In Panel A, we work with levels of CIQ(τ) factors and other factors, in Panel B, we focus on differences of the factors. First, we look at the dependence between CIQ(τ) factors and PCA-SQ factor. We can see that the correlation is the strongest if we move to the tails with the correlation for CIQ(0.1) and PCA-SQ being equal to -0.76 for the level of the factors but it decreases substantially if we look at the differences with the correlation being equal to -0.53. Moreover, the correlation is stronger for the CIQ(τ) factors with τ above the median.

[10] If not stated otherwise, in the rest of the paper, we perform all the analyses using ΔCIQ(τ) factors.

Next, we look at the correlations with the common idiosyncratic variance factor of Herskovic et al. (2016). In this case, the correlations are slightly higher for τs below the median, with peak correlation at $\tau = 0.1$ being equal to -0.45 for the levels of the factors. On the other hand, if we move to the differences, the correlation decreases to -0.21. Correlations with the tail risk factor (TR) are relatively small with the peak at $\tau = 0.8$ with a -0.29 correlation. Especially low are the correlations between TR factor and CIQ(τ) factors for downside values of τ. Correlations with the variance risk premium (VRP) factor of Bollerslev et al. (2009) are very low as well, with values no higher than 0.12 in absolute value for both levels and differences of the factors. Finally, correlations with the VIX index are symmetrical around the median τ with a peak of 0.39 at $\tau = 0.9$ while there is a stronger correlation between downside τs and the VIX with values around 0.26 in differences.

This preliminary analysis suggests that behavior of idiosyncratic quantiles shocks is in non-negligible part distinct from shocks to volatility and downside risk measures.

In addition, Table 2 provides correlations between CIQ(τ) factors at different quantiles. Correlation between CIQ(τ) in levels for the upper and lower part of the distribution are far from perfect, e.g., the correlation between the lower tail factor CIQ(0.1) and upper tail CIQ(0.9) is -0.69. This observation suggests that the factors do not simply duplicate information and are hence not likely to be rescaled information contained in common volatility factor (captured by e.g., PCA-SQ). Moreover, this dependence decreases substantially if we look at the increments of the CIQ(τ) factors – dependence between lower and upper tail factors reduces to -0.32. These results suggest that there is a potential for different pricing information across quantiles and that this information does not simply mirror information contained in the common volatility.

Table 2: *Correlations between CIQ(τ) factors.* The table presents unconditional correlations between CIQ(τ) factors in levels (above diagonal) and differences (below diagonal). We estimate the factors using FF3 residuals of the monthly CRSP stocks' returns. Data contain the period between January 1963 and December 2018.

τ	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9
0.1	.	0.98	0.95	0.86	0.55	-0.03	-0.05	-0.32	-0.56	-0.63	-0.69
0.15	0.97	.	0.98	0.91	0.63	0.00	0.01	-0.24	-0.50	-0.58	-0.65
0.2	0.93	0.97	.	0.95	0.71	0.05	0.05	-0.16	-0.42	-0.52	-0.60
0.3	0.85	0.91	0.95	.	0.82	0.13	0.15	0.06	-0.22	-0.33	-0.43
0.4	0.68	0.77	0.83	0.93	.	0.23	0.28	0.36	0.12	0.02	-0.08
0.5	0.07	0.12	0.17	0.25	0.34	.	0.75	0.41	0.34	0.30	0.26
0.6	0.12	0.17	0.21	0.29	0.40	0.78	.	0.47	0.40	0.37	0.32
0.7	0.14	0.24	0.31	0.49	0.66	0.47	0.54	.	0.93	0.87	0.79
0.8	-0.10	-0.01	0.07	0.25	0.46	0.41	0.48	0.92	.	0.98	0.94
0.85	-0.21	-0.13	-0.05	0.13	0.35	0.39	0.46	0.85	0.96	.	0.97
0.9	-0.32	-0.25	-0.18	-0.01	0.22	0.33	0.39	0.75	0.90	0.95	.

Overall, we can see that the correlations between CIQ(τ) factors and other related factors are far from perfect. The highest degree of comovement is, not surprisingly, seen for levels of CIQ(τ) factors and PCA-SQ factor, which is substantially reduced if we look at the differences of those factors. Moreover, a strong asymmetry in the correlations across τ suggests that the information contained in the downside and upside CIQ factors differ.

3 Time-series Predictability of Market Return

We start examining the information content of the CIQ(τ) factors for subsequent short-term market returns. Here we aim to predict the monthly excess return on the market that we approximate by the value-weighted return of all CRSP firms. In the regressions, we also control for popular predictive variables used in Welch and Goyal (2007) as well as three closely related factors – TR factor of Kelly and Jiang (2014), the innovations of common idiosyncratic volatility (ΔCIV) factor of Herskovic et al. (2016), and the VRP factor of Bollerslev et al. (2009).[11] Moreover, we construct the PCA-SQ factor and use its increments to control for the effect of the common volatility. Because the CIQ(τ) factors are estimated using a rolling window, we use the last value of the factors estimated from each rolling window to construct a single series of the CIQ(τ) factors.

First, we report the results from the univariate regressions of the market return on the differences of the CIQ(τ) factors at various τ quantile levels of the form

$$r_{m,t+1} = \gamma_0 + \gamma_1 \times \Delta f_t(\tau) + \epsilon_{t+1} \tag{13}$$

in Table 3. We report estimated scaled coefficients to capture the effect of one standard deviation increase of the independent variable on the subsequent annualized market return. The corresponding t-statistics are computed using Newey-West robust standard errors using six lags.

The results in Table 3 document strong predictive power using the ΔCIQ(τ) factors for the left part of the distribution, with the peak for $\tau = 0.3$, where the increase (decrease) of one standard deviation in the factor predicts subsequent decrease (increase) of 7.05 percents in annualized market return.[12] There is also some predictive power for the upper tail factor when CIQ(0.9), but the effect is much smaller with only 3.50 percent increase in annualized market return accompanied with only less than one-third of the R^2 from the lower tail. From

[11]We replicated tail risk factor construction of Kelly and Jiang (2014) by ourself; we acquired data of Herskovic et al. (2016) from Bernard Herskovic's webpage and data of Bollerslev et al. (2009) from Hao Zhou's webpage.

[12]Note that the lower tail factors are on average negative. Increase (decrease) of these factors corresponds to the decrease (increase) of risk, which leads to a decrease (increase) of the required risk premium.

Table 3: *Predictive power of the ΔCIQ(τ) factors.* The table reports results from the univariate predictive regressions of the value-weighted return of all CRSP firms on the ΔCIQ(τ) factors for various $\tau \in (0, 1)$. Coefficients are scaled to capture the effect of one standard deviation increase in the factor on the annualized market return in percent. The corresponding t-statistics are computed using the Newey-West robust standard errors using six lags. We report both in-sample (IS) and out-of-sample (OOS) R^2s. We also truncate the predictions at zero following Campbell and Thompson (2007) (CT) and report corresponding IS and OOS R^2s. The time span covers the period between January 1960 and December 2018.

τ	Coeff.	t-stat	R^2 IS	R^2 OOS	R^2 IS CT	R^2 OOS CT
0.1	-6.31	-2.77	1.40	1.09	1.21	1.42
0.15	-6.49	-2.74	1.48	1.17	1.20	1.45
0.2	-6.38	-2.63	1.43	1.13	1.14	1.33
0.3	-7.05	-2.98	1.75	1.21	1.21	1.41
0.4	-6.59	-2.92	1.53	0.58	0.83	0.76
0.5	0.15	0.07	0.00	-0.37	0.00	-0.19
0.6	0.29	0.13	0.00	-0.30	0.00	-0.23
0.7	-0.88	-0.48	0.03	-0.67	0.03	-0.37
0.8	2.09	1.13	0.15	-0.26	0.10	-0.08
0.85	3.05	1.67	0.33	-0.03	0.21	0.31
0.9	3.50	1.88	0.43	0.06	0.29	0.31

a perspective of an investor, in times of high risk – captured by large negative increments of the left-tail CIQ(τ) factor, she requires a premium for investing. And thus, these risky periods correlate with the high marginal utility states of the investors.

Together with in-sample (IS) R^2, we also report the out-of-sample (OOS) R^2 from expanding window scheme. We use data up to time t to estimate the prediction model and then forecast the $t + 1$ return (the first window contains 120 monthly periods to obtain sufficiently reasonable estimates). Then, the window is extended by one observation, the prediction model is re-estimated and a new forecast is obtained. We repeat this procedure until the whole sample is exhausted. The corresponding R^2 is computed by comparing conditional forecast and historical mean computed using the available data up to time t, i.e., $1 - \sum_t (r_{m,t+1} - \widehat{r}_{m,t+1|t})^2 / \sum_t (r_{m,t+1} - \bar{r}_{m,t})^2$ where $\widehat{r}_{m,t+1|t}$ is out-of-sample forecast of the $t + 1$ return using data up to time t, and $\bar{r}_{m,t}$ is the historical mean of the market return computed up to date t. Unlike the case of the IS R^2, the OOS R^2 can attain negative values if the conditional forecasts perform worse than the historical mean forecast. The positive values of the OOS R^2 for τ between 0.1 and 0.4 provide strong evidence for the benefits of the ΔCIQ(τ) factors for predicting the market return in the real-world setting. On the other hand, the predictability vanishes for the higher values of τ.

To assess the economic usefulness for the investors, we further follow suggestions from Campbell and Thompson (2007) (hence CT). They propose to truncate the predictions from the estimated model at 0, as the investor would not have used a model to predict a negative premium. This non-linear modification of the model should introduce caution into the models. Based on this modification, we report both IS and OOS R^2s. Naturally, using this transformation, the IS R^2 does not improve for any of the models, but the performance rises for the OOS analysis. Results suggest that the common fluctuations in the lower part of the excess returns distributions robustly predict the subsequent market movement.

Next, we run bivariate regressions to assess whether the proposed quantile factors contain additional information not included in the relevant previously proposed variables

$$r_{m,t+1} = \gamma_0 + \gamma_1 \times \Delta f_t(\tau) + \gamma_2 \times f_t^{Control} + \epsilon_{t+1} \tag{14}$$

where we separately control for variables that may contain duplicate information. First, in Table 4, we report coefficients and their t-statistics while controlling for differences of the PCA-SQ factor, the ΔCIV of Herskovic et al. (2016), the TR factor of Kelly and Jiang (2014), and the VRP factor of Bollerslev et al. (2009), respectively. For better comparability, we also include results from the univariate predictions using the $\Delta\text{CIQ}(\tau)$ factors only. In the case of PCA-SQ factor, we can see that neither the significance nor the magnitude of the predictive power of the downside CIQ factors is diminished. Moreover, the borderline significance of the upside CIQ factors vanishes. This suggests that the common volatility element is not the driving force of the predictive performance of the quantile factors. In the second case, while controlling for the ΔCIV, the results regarding the $\Delta\text{CIQ}(\tau)$ factors remain the same, and ΔCIV proves not to predict future market returns. In the case of the TR factor, the $\Delta\text{CIQ}(\tau)$ factors mirror the results from the univariate regressions in terms of coefficients and their significance. TR factor is significant across all the specifications, although its effect is smaller and less significant than in the case of $\Delta\text{CIQ}(\tau)$ for the lower tail values of τ. In the third case, the VRP factor appears to be the most closely related in terms of predictability to the $\Delta\text{CIQ}(\tau)$ factors.[13] The VRP is highly significant, and at the same time, it diminishes the effect of the $\Delta\text{CIQ}(\tau)$ factors – the scaled coefficients decreases around 1.7 percentage points, and the corresponding t-statistics are now approximately 1.5. This decrease in significance may be also caused by substantial decrease of the available time period as the VRP starts in 1990.

As a next step, we control for variables discussed in Welch and Goyal (2007).[14] Instead of a large table of coefficients and t-statistics through all variables and quantiles, we summarize the results in the Table 5, in which we include t-statistics of the $\Delta\text{CIQ}(\tau)$ factors from the bivariate regressions of the form 14 while controlling for said variables. We observe that none

[13]We acknowledge that there is no clear theoretical link between VRP and $\Delta\text{CIQ}(\tau)$ factors. The VRP is associated with the aggregate S&P 500 composite index (rather than the value-weighted return of all CRSP stocks), which it only significantly predicts over a medium-term horizon. However, we have included it for informational purposes and to potentially stimulate a discussion regarding the relationship between these two phenomena in the future.

[14]For the information regarding the specification of the variables, see Welch and Goyal (2007). We obtained the data from the Iwo Welch's webpage.

Table 4: *Bivariate predictive regressions.* The table reports results from the bivariate predictive regressions of the value-weighted return of all CRSP firms on ΔCIQ(τ) factors for various $\tau \in (0,1)$ and other control variables. We employ the PCA-SQ factor, innovations of CIV factor of Herskovic et al. (2016), TR factor of Kelly and Jiang (2014), and the VRP factor of Bollerslev et al. (2009), respectively. Coefficients are scaled to capture the effect of one-standard-deviation increase in the factor on the annualized market return in percent. The corresponding t-statistics are computed using the Newey-West robust standard errors using six lags. The time span covers the period between January 1960 and December 2018 except the VRP that starts in January 1990.

control / τ	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9
CIQ	-6.31	-6.49	-6.38	-7.05	-6.59	0.15	0.29	-0.88	2.09	3.05	3.50
	(-2.77)	(-2.74)	(-2.63)	(-2.98)	(-2.92)	(0.07)	(0.13)	(-0.48)	(1.13)	(1.67)	(1.88)
R^2	1.40	1.48	1.43	1.75	1.53	0.00	0.00	0.03	0.15	0.33	0.43
CIQ	-6.05	-6.12	-5.89	-6.54	-6.33	-0.62	-0.52	-2.56	0.22	1.35	1.93
	(-2.45)	(-2.43)	(-2.35)	(-2.83)	(-2.83)	(-0.31)	(-0.24)	(-1.21)	(0.10)	(0.56)	(0.78)
PCA-SQ	0.48	0.77	1.14	1.74	3.14	3.79	3.77	4.59	3.54	2.86	2.40
	(0.22)	(0.36)	(0.56)	(0.91)	(1.59)	(1.96)	(1.90)	(2.00)	(1.40)	(1.05)	(0.86)
R^2	1.40	1.50	1.47	1.85	1.87	0.48	0.48	0.67	0.47	0.51	0.55
CIQ	-6.72	-6.89	-6.71	-7.29	-6.71	0.18	0.33	-0.84	2.17	3.15	3.56
	(-2.82)	(-2.77)	(-2.66)	(-2.98)	(-2.87)	(0.08)	(0.15)	(-0.47)	(1.14)	(1.68)	(1.87)
ΔCIV	-1.96	-1.96	-1.78	-1.61	-1.19	-0.57	-0.58	-0.49	-0.77	-0.92	-0.84
	(-0.59)	(-0.59)	(-0.54)	(-0.49)	(-0.36)	(-0.16)	(-0.17)	(-0.14)	(-0.22)	(-0.26)	(-0.24)
R^2	1.53	1.61	1.54	1.84	1.58	0.01	0.01	0.04	0.17	0.36	0.45
CIQ	-6.28	-6.44	-6.36	-6.99	-6.52	0.31	0.35	-0.76	2.27	3.12	3.58
	(-2.76)	(-2.72)	(-2.63)	(-2.96)	(-2.88)	(0.15)	(0.16)	(-0.41)	(1.22)	(1.72)	(1.93)
TR	4.67	4.64	4.69	4.62	4.60	4.72	4.71	4.69	4.80	4.76	4.77
	(2.33)	(2.32)	(2.35)	(2.31)	(2.31)	(2.33)	(2.33)	(2.32)	(2.35)	(2.34)	(2.34)
R^2	2.17	2.24	2.20	2.50	2.27	0.78	0.78	0.80	0.96	1.12	1.23
CIQ	-4.63	-4.80	-4.60	-4.67	-4.54	0.41	0.43	-0.92	0.85	2.31	1.67
	(-1.39)	(-1.38)	(-1.38)	(-1.46)	(-1.47)	(0.15)	(0.14)	(-0.35)	(0.33)	(0.87)	(0.67)
VRP	11.83	11.79	11.62	11.55	11.44	11.58	11.56	11.52	11.64	11.73	11.67
	(5.62)	(5.60)	(5.38)	(5.31)	(5.22)	(5.35)	(5.32)	(5.33)	(5.45)	(5.54)	(5.50)
R^2	6.06	6.12	6.05	6.07	6.03	5.23	5.23	5.25	5.25	5.43	5.33

of the variables drives out the significance of the ΔCIQ(τ) factors. Moreover, the magnitude of the significance remains very close to the ones from the univariate regressions.

3.1 Prediction using many CIQ(τ) Factors

Because it is *ex-ante* not clear on which quantile the investor should base her investment strategy on, we perform an out-of-sample prediction exercise which utilizes information from more than one ΔCIQ(τ) factor when constructing a forecast. The results are summarized in Table 6. We use either all of the factors when predicting the market return or we use two disjunct subsets of them. Using the first subset, we employ a prior assumption that only the downside factors ($\tau < 0.5$) are significant predictors of the market return. Second subset imposes the premise that the upside factors ($\tau > 0.5$) possess the forecasting power for the aggregate return. To do that, we use various models to exploit the information from the ΔCIQ(τ) factors. We train the models on the first 120 monthly observations and then

Table 5: *Controlled predictive significance of the ΔCIQ(τ) factors using Welch and Goyal (2007) variables.* The table summarizes t-statistics associated with the ΔCIQ(τ) factors from bivariate regressions when controlling for macroeconomic variables discussed in Welch and Goyal (2007). The dependent variable is the value-weighted return of all CRSP firms. The t-statistics are computed using the Newey-West robust standard errors using six lags. The time span covers the period between January 1960 and December 2018.

control/ τ	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9
dp	-2.78	-2.75	-2.65	-3.01	-2.94	0.04	0.11	-0.51	1.11	1.65	1.88
dy	-2.75	-2.72	-2.63	-2.98	-2.92	0.04	0.11	-0.52	1.09	1.61	1.84
ep	-2.77	-2.74	-2.64	-2.99	-2.93	0.06	0.13	-0.52	1.11	1.66	1.87
de	-2.77	-2.74	-2.63	-2.98	-2.91	0.07	0.12	-0.46	1.16	1.71	1.90
svar	-2.81	-2.75	-2.64	-2.96	-2.87	0.10	0.21	-0.23	1.39	1.87	2.06
bm	-2.77	-2.74	-2.63	-2.98	-2.92	0.07	0.13	-0.49	1.12	1.67	1.88
ntis	-2.72	-2.69	-2.59	-2.93	-2.89	0.07	0.13	-0.47	1.12	1.67	1.87
tbl	-2.75	-2.74	-2.62	-2.95	-2.89	0.08	0.16	-0.44	1.15	1.71	1.88
lty	-2.75	-2.73	-2.61	-2.96	-2.89	0.07	0.15	-0.46	1.13	1.68	1.88
ltr	-2.52	-2.52	-2.44	-2.82	-2.79	-0.03	0.08	-0.50	0.94	1.47	1.63
tms	-2.82	-2.79	-2.68	-3.03	-2.97	0.09	0.14	-0.46	1.19	1.71	1.91
dfy	-2.72	-2.69	-2.59	-2.95	-2.92	0.09	0.13	-0.47	1.11	1.62	1.82
infl	-2.63	-2.61	-2.50	-2.85	-2.84	0.14	0.17	-0.45	1.08	1.62	1.79

expand the estimation window as discussed before. We report both simple OOS R^2 and OOS R^2 CT to asses the fit. When performing regularization in the coefficient estimation, one has to choose so called tuning parameters. We choose the tuning parameters based on the in-sample leave-one-out full cross-validation procedure. We chose the forecast construction methods following Dong et al. (2022).

The first models that we employ is an OLS model which uses a OLS fitted multivariate regression model (estimated in-sample) to predict one-month-ahead return of the market. We can see that using all the ΔCIQ(τ) factors to predict OOS return yields a negative R^2. This is caused by the overfitting problem when we use many correlated variables and do not impose any parameter regularization. Using only either downside or upside factors and truncating the prediction at zero, yield some marginal gains for the investor.

The LASSO (least absolute shrinkage and selection operator, Tibshirani (1996)) model (estimator) introduces a regularization in the estimation procedure of the predictive coefficients. In the case of LASSO, only a subset of the predictors is chosen to have non-zero coefficients. As we can see, the performances for all τ and downside τ models substantially improve. On the other hand, prediction based on the upside τs do not yield a good fit even after the introduction of a regularization.

Next, we generalize the previous LASSO model and report results based on the elastic net (ENET) estimator (Zou and Hastie, 2005). The estimator employs ℓ_1 (LASSO) and ℓ_2 (ridge regression, Hoerl and Kennard (1970)) penalty terms. For simplicity reasons, we chose the penalty weights to be both equal to 0.5 without any tuning procedure. As we can see, the results closely mirror the results from the LASSO estimation.

As a next model, we perform a simple combination forecast. We first obtain univariate

Table 6: *Out-of-sample performance of the forecast combinations.* The table reports performance of various specifications of multivariate predictive models using all ΔCIQ(τ) factors, τ below median ΔCIQ factors (downside), or above median ΔCIQ factors (upside). The time span covers the period between January 1960 and December 2018.

	All τ		Downside τ		Upside τ	
model	R^2	R^2 CT	R^2	R^2 CT	R^2	R^2 CT
OLS	-1.53	-0.40	-0.44	0.53	-0.31	0.39
LASSO	0.94	0.95	0.21	0.80	-0.25	0.14
ENET	0.92	1.03	0.07	0.71	-0.11	0.27
Combination	1.10	1.06	1.26	1.39	-0.07	0.07
C-LASSO	0.79	0.92	0.93	1.29	-0.61	-0.23
C-NET	0.86	0.78	0.85	1.22	-0.65	-0.19
PCA	1.17	1.22	1.21	1.46	-0.31	-0.10
OLS selection	0.87	1.28	0.87	1.28	-0.73	-0.10

forecasts for each ΔCIQ(τ) factor separately and then the final forecast is obtained as a simple average of the univariate forecasts. We can see that the model performs very well for selection of all τs and downside τs, with R^2 being up to 1.26% for downside τs and R^2 CT of 1.39%. On the other hand, upside τs do not lead to any valuable forecasts.

C-LASSO and C-NET follow the same idea as the Combination model but instead of averaging all the univariate forecasts, they run multivariate penalized regression (LASSO and ENET, respectively) of the future market return on the univariate forecasts to select the best combination of them. The resulting forecast is then obtained by plugging the last value of ΔCIQ(τ) from a window into the fitted models. Once again, all τ and downside τ subsets perform both very well, with R^2 of 0.93% and R^2 CT of 1.29% for downside τ C-LASSO. But the models using upside τ yield even negative R^2. This is the case for all the remaining models which use upside τ factors.

PCA model aggregates information and creates the first principal component from all the ΔCIQ(τ) factors and uses it as the prediction variable in the univariate prediction regression. We observe that the downside τ PCA model performs the best across all the specifications.

Finally, the OLS selection model fits univariate prediction models for each ΔCIQ(τ) factor and uses the univariate model 13 with the best in-sample fit to predict the future market return. This simple approach yields very solid performance of 0.87% for R^2 and 1.28% for R^2 CT.

To summarize this section, we observed that using the downside ΔCIQ(τ) factors in various multivariate models, we obtain significant positive performance. On the contrary, the upside ΔCIQ(τ) factors do not result into economic gains because they do not outperform the forecasts based on the historical mean. All the results thus suggest that the driving force behind the downside quantile factors' performance is not the common volatility component but the information contained in the left part of the common factor structure.

4 Pricing the CIQ(τ) Risks in the Cross-Section

In this section, we investigate the pricing implications of the presented common idiosyncratic quantile factors for the cross-section of stock returns. We hypothesize that the stochastic discount factor increases in the CIQ(τ) risk, as the risk-averse investor's marginal utility is high in the states of high CIQ(τ) risk. Based on that hypothesis, we assume that the assets that perform poorly in the states of high CIQ(τ) risk will require a higher risk premium for holding by the investors. On the other hand, assets that perform well during these states serve as a hedging tool and will be traded with higher prices and thus lower expected returns. The stock's sensitivities to the factors capture betas estimated by the linear regression of stocks' returns on the factors. If not explicitly stated otherwise, we use as our predicted variable monthly out-of-sample returns following the estimation window. We also try to predict one-year returns using portfolios to assess the persistence of the CIQ(τ) betas and thus indirectly investigate the transaction costs related to the trading of these factors. Data that we employ cover the usual asset pricing period between January 1963 and December 2018. We exclude "penny stocks" with prices less than one dollar to avoid related biases.

To alleviate the concerns that the quantile factors simply mirror the dynamics of the idiosyncratic volatilities of the single-stock returns, in the case of pricing the cross-section, we perform the estimation of the factors using standardized idiosyncratic returns.[15] Specifically, we estimate time-varying volatility using exponentially weighted moving average model. Then, we use the $\Delta f_t(\tau)$ estimates as our risk factors. For all available stocks and and for all τ, we estimate quantile-specific betas

$$r_{i,t} = \alpha_i + \beta_i(\tau)\Delta f_t(\tau) + v_{i,t}(\tau),$$

using the least-square estimator. These betas will be used in the following asset pricing tests as a measure of the exposure to the CIQ(τ) factors. Same as the factors, betas are also estimated using the 60-month rolling window. We include the stocks that possess at least 48 monthly observations. Betas computed up to time t are used to predict returns at time $t+1$ or further – no overlap between estimation and prediction periods. The control variables are estimated using the same procedure as originally proposed.

Later in the analysis, we also control for the effect of the increments of the PCA-SQ factor, ΔCIV factor and many other related variables to show that the effect of the newly proposed quantile factors is not subsumed by the effect of any related factor or stock-specific variable.

[15] In Appendix in Table 16, we report correlations between the CIQ(τ) factors estimated using standardized data. The correlations are generally smaller.

4.1 Cross-sectional Regressions

As a first step in the investigation of the cross-sectional implications of exposures to the common idiosyncratic quantile risks, we perform two-stage Fama and MacBeth (1973) predictive regressions. We explore the hypothesis that the exposures to the ΔCIQ(τ) factors align with the future excess returns of the stocks. This type of asset pricing test moreover conveniently allows for simultaneous estimation of many risk premiums associated with various risk measures. That means that we can estimate the risk premium associated with the CIQ(τ) risks while controlling for other risk measures previously proposed in the literature. More specifically, for each time $t = 1, \dots, T-1$ using all of the stocks $i = 1, \dots, N$ available at time t and $t+1$,[16] we cross-sectionally regress all the returns at time $t+1$ on the betas estimated using only the information available up to time t. This procedure yields estimates of prices of risk $\lambda_{t+1}(\tau)$ while controlling for the most widely used competing measure of risk

$$r_{i,t+1} = \alpha + \beta_{i,t}^{CIQ(\tau)}(\tau)\lambda_{t+1}^{CIQ(\tau)}(\tau) + \beta_{i,t}^{\top Control}\lambda_{t+1}^{Control} + e_{i,t+1} \tag{15}$$

where $\beta_{i,t}^{Control}$ is vector of control betas or other stock characteristics and $\lambda_{t+1}^{Control}$ is vector of corresponding prices of risk. Using $T-1$ cross-sectional estimates of the prices of risk, we compute the average price of risk associated with each $\lambda^{CIQ}(\tau)$ as

$$\widehat{\lambda}^{CIQ(\tau)}(\tau) = \frac{1}{T-1}\sum_{t=2}^{T}\widehat{\lambda}_t^{CIQ(\tau)}(\tau) \tag{16}$$

and report them along with their t-statistics based on the Newey-West robust standard errors.

We summarize the first set of results in Table 7 where we report estimation outcomes of controlling the effect of ΔCIQ(τ) factors by general risk measures. But first, we report results from the univariate regressions on CIQ(τ) betas. We observe similar results to those obtained from the market predictions – the exposure to the common idiosyncratic downside events is significantly compensated in the cross-section of stock returns. For example, CIQ(τ) for $\tau = 0.2$ possess a coefficient of 1.11 (t-stat = 2.57), on the other hand, for $\tau = 0.8$, the estimated coefficient is equal to -0.14 (t-stat = -0.30). This suggests that the exposure to the common idiosyncratic downside events is significantly compensated in the cross-section. On the contrary, to hold assets with high exposure to the upside common movements the investors have to pay a small discount for those stock, although not statistically significant one.

[16] A stock is identified as available, if it possess at least 48 monthly return observations during the last 60-month window up to time t and also an observation at time $t+1$.

Table 7: *Fama-MacBeth regressions using $\Delta CIQ(\tau)$ factors and general risk measures.* The table contains estimated prices of risk and t-statistics from the Fama-MacBeth predictive regressions. Each segment contains prices of risk of CIQ(τ) betas while controlling for various risk measures. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$. Note the coefficients are multiplied by 100 for clarity of presentation.

	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9
CIQ(τ)	0.90	0.94	1.11	1.52	2.50	0.55	1.01	0.71	-0.14	-0.22	-0.27
	(2.52)	(2.42)	(2.57)	(2.88)	(2.80)	(0.20)	(0.41)	(1.13)	(-0.30)	(-0.49)	(-0.60)
CIQ(τ)	0.41	0.46	0.59	0.95	1.78	0.55	0.91	1.02	0.39	0.30	0.22
	(1.43)	(1.53)	(1.76)	(2.18)	(2.25)	(0.21)	(0.39)	(1.74)	(0.97)	(0.80)	(0.59)
Mkt	-0.23	-0.23	-0.23	-0.22	-0.23	-0.25	-0.25	-0.24	-0.25	-0.25	-0.24
	(-1.70)	(-1.72)	(-1.71)	(-1.68)	(-1.69)	(-1.88)	(-1.87)	(-1.77)	(-1.83)	(-1.85)	(-1.84)
CIQ(τ)	0.72	0.75	0.87	1.19	2.03	-0.37	-1.11	0.41	-0.08	-0.11	-0.14
	(2.56)	(2.47)	(2.54)	(2.64)	(2.65)	(-0.17)	(-0.52)	(0.80)	(-0.22)	(-0.34)	(-0.42)
Idiosyncratic volatility	-14.12	-14.14	-14.15	-14.17	-14.40	-14.57	-14.61	-14.64	-14.58	-14.24	-13.86
	(-2.18)	(-2.18)	(-2.17)	(-2.20)	(-2.17)	(-2.20)	(-2.20)	(-2.19)	(-2.21)	(-2.18)	(-2.12)
CIQ(τ)	0.87	0.91	1.08	1.47	2.43	0.42	0.91	0.72	-0.13	-0.22	-0.28
	(2.47)	(2.38)	(2.53)	(2.76)	(2.74)	(0.15)	(0.38)	(1.17)	(-0.29)	(-0.50)	(-0.63)
Idiosyncratic skewness	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02
	(0.26)	(0.29)	(0.28)	(0.31)	(0.33)	(0.37)	(0.41)	(0.47)	(0.47)	(0.53)	(0.57)
CIQ(τ)	0.86	0.90	1.07	1.47	2.43	0.43	0.91	0.73	-0.12	-0.21	-0.27
	(2.46)	(2.37)	(2.52)	(2.76)	(2.75)	(0.16)	(0.38)	(1.18)	(-0.27)	(-0.48)	(-0.61)
Skewness	-0.38	-0.30	-0.34	-0.25	-0.17	-0.13	-0.03	0.17	0.15	0.32	0.46
	(-0.13)	(-0.10)	(-0.12)	(-0.09)	(-0.06)	(-0.04)	(-0.01)	(0.06)	(0.05)	(0.11)	(0.17)

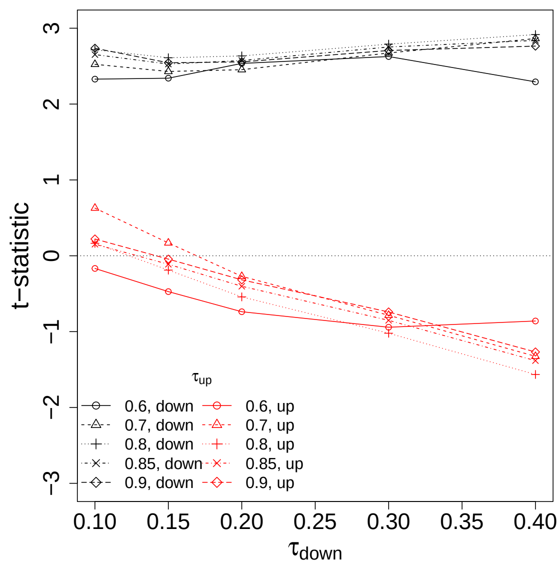
As those results suggest, there is a strong asymmetry in the pricing implications of the ΔCIQ(τ) factors. To further assess it, we perform the following set of bivariate regressions

$$r_{i,t+1} = \alpha_{t+1} + \beta_{i,t}^{CIQ}(\tau_{down})\lambda_{t+1}(\tau_{down})^{CIQ} + \beta_{i,t}^{CIQ}(\tau_{up})\lambda_{t+1}(\tau_{up})^{CIQ} + e_{i,t+1},$$
$$\tau_{down} = \{0.1, 0.15, 0.2, 0.3, 0.4\}, \tau_{up} = \{0.6, 0.7, 0.8, 0.85, 0.9\} \quad (17)$$

where we assess the joint effect of downside and upside CIQ(τ) factors. We report t-statistics for each pair of $\lambda(\tau_{down})^{CIQ}$ and $\lambda(\tau_{up})^{CIQ}$ in the Figure 1. We observe that the prices of risk associated with downside risk remain statistically significant using every combination of downside and upside CIQ factors. On the other hand, the risk prices for the upside potential are in agreement with the previous results – insignificant but negative when controlling for higher values of τ_{down}.

Next, in the rest of the Table 7, we present results from bivariate regressions when controlling for the effect of general risk measures. We report the results of including CAPM betas by regressing the returns on the market return (Mkt). Interestingly, the effect of the CAPM beta diminishes the pricing relationship for the extreme left τ CIQ factors but the price of risk related to the linear exposure to the market factor possess counterintuitive negative sign – consistent with previous empirical evidence. Next, we control for the effect of the idiosyncratic volatility computed from the residuals of the 3-factor model of Fama

Figure 1: $\Delta CIQ(\tau)$ *betas – bivariate cross-sectional regressions.* The figure reports t-statistics of prices of risks from bivariate regressions from the Equation 17 of CIQ(τ) betas for downside and upside τs. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$.



and French (1993). The effect of the CIQ exposures remain very close to the one from the univariate regressions. Besides that, we confirm the presence of the idiosyncratic volatility puzzle. Next, we present results when controlling for idiosyncratic and total skewness. Those variables do not possess a significant pricing information for the cross-section, on the other hand, the effect of the CIQ factors remain consistent with the previous results.

Second, we report results from the bivariate[17] regressions in which we include as a control various risk measures based on common volatility or asymmetric dependence. Those measures were previously in the literature proven to be significant predictors of expected returns. We summarize the estimation outcomes in Table 8.

To investigate whether the quantile factors provide different priced information beyond conventional approximate factor models, we construct and control for the following factor related to the common volatility. To do that, we proceed similarly as in the case of market prediction and construct a factor based on principal component analysis that captures dynamics in the common volatility. More specifically, as in the construction of the quantile factors – using the 60-month moving window, we extract the standardized idiosyncratic returns and square them. Then, we perform principal component analysis on those squared residuals and take the first principal component that explains the most common time variation across the squared residuals, and we denote it as PCA-SQ. We then difference the

[17]Except for the coskewness and cokurtosis, which we include both at the same time in the regression.

factor and use its increments as a control factor. From the results, we can conclude that the quantile factors extract very different information regarding the expected returns, as the specification based on the factor extracted from the squared residuals turn out not to be a significant predictor in the cross-section of stock returns. One has to look deeper into the common distribution if he wants to identify priced information regarding the common distributional movements.

Next, we employ volatility betas computed on differences of the CIV factor of Herskovic et al. (2016). We see that the results regarding CIQ(τ) betas still hold both qualitatively and quantitatively similar to the case of univariate regressions. Moreover, CIV risk is priced as well; especially strong is the relationship when we control for CIQ(τ) betas with τ from the right part of the distribution. These results suggest that both common idiosyncratic volatility and quantile risk are priced and do not convey the same pricing information.

As another related control, we use the tail risk (TR) factor of Kelly and Jiang (2014). As we can see, TR betas do not drive out the CIQ(τ) betas' effect, which remains significant, similarly to the univariate specification. Next, we control for related group of risk measures which consider the non-linear relationship between asset and market returns. By following the specifications of Harvey and Siddique (2000) and Ang et al. (2006), respectively, we control simultaneously for coskewness and cokurtosis. Once again, those measures do not drive out the significance of the CIQ(τ) betas. Coskewness possess the expected sign but it is not statistically significant. On the other hand, cokurtosis is borderline significant for $\tau \geq 0.5$ but with opposite sign than expected.

Another approach to capture non-linear dependence is via downside risk (DR) beta, which describes conditional covariance below some threshold level. We entertain the specification of Ang et al. (2006), which sets the threshold value equal to the average market return. As we can see, downside beta do not subsume the effect of the ΔCIQ(τ) factors, neither it is a significant predictor of future returns.

Another related left-tail risk measure is hybrid tail covariance risk (HTCR) measure proposed by Bali et al. (2014). Although, it is highly significant predictor of expected returns, it does not drive the effect of the CIQ(τ) risks out. Next, we include negative semibeta (β^{-}) of Bollerslev et al. (2021) in our bivariate regression. Similarly as in the previous cases, the exposure to the quantile factors yields a significant risk premium.

Then, to control for the effect of comovement asymmetry between left and right parts of the joint distribution of stock and market return, we include downside asymmetric comovement (DOWN ASY) measure of Jiang et al. (2018). This measure does not affect the relationship between expected returns and CIQ(τ) betas either.

To control for the effect of crashes in many risk factors, we control for multivariate crash

Table 8: *Fama-MacBeth regressions using CIQ(τ) factors and asymmetric risk measures.* The table contains estimated prices of risk and t-statistics from the Fama-MacBeth predictive regressions. Each segment contains prices of risk of ΔCIQ(τ) betas while controlling for various asymmetric risk measures. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$. Note the coefficients are multiplied by 100 for clarity of presentation.

	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9
CIQ(τ)	0.87	0.93	1.01	1.31	2.21	0.40	0.79	1.09	0.20	0.20	-0.12
	(2.45)	(2.52)	(2.53)	(2.69)	(2.42)	(0.14)	(0.30)	(1.55)	(0.43)	(0.43)	(-0.26)
PCA-SQ	8.56	8.97	2.57	-9.04	-22.35	-23.82	-26.22	-39.26	-29.58	-30.30	-14.48
	(0.35)	(0.37)	(0.11)	(-0.38)	(-0.92)	(-0.93)	(-1.01)	(-1.38)	(-1.04)	(-1.07)	(-0.54)
CIQ(τ)	0.77	0.81	0.99	1.38	2.34	1.26	2.14	0.89	-0.04	-0.13	-0.18
	(2.09)	(2.10)	(2.38)	(2.66)	(2.68)	(0.49)	(0.91)	(1.61)	(-0.10)	(-0.30)	(-0.41)
CIV	-0.39	-0.40	-0.43	-0.46	-0.50	-0.57	-0.55	-0.55	-0.53	-0.53	-0.51
	(-1.58)	(-1.64)	(-1.75)	(-1.91)	(-2.08)	(-2.46)	(-2.35)	(-2.34)	(-2.21)	(-2.23)	(-2.16)
CIQ(τ)	0.86	0.88	1.03	1.37	2.16	-0.77	-0.24	0.24	-0.43	-0.44	-0.44
	(2.45)	(2.28)	(2.42)	(2.64)	(2.53)	(-0.29)	(-0.10)	(0.41)	(-0.91)	(-1.00)	(-1.00)
TR	0.11	0.11	0.11	0.12	0.12	0.12	0.11	0.12	0.12	0.12	0.12
	(1.33)	(1.30)	(1.32)	(1.40)	(1.41)	(1.47)	(1.42)	(1.38)	(1.43)	(1.42)	(1.36)
CIQ(τ)	0.82	0.87	1.03	1.41	2.25	0.28	0.89	0.82	-0.03	-0.12	-0.18
	(2.40)	(2.39)	(2.47)	(2.83)	(2.76)	(0.11)	(0.39)	(1.38)	(-0.07)	(-0.27)	(-0.41)
Coskew	-0.12	-0.13	-0.14	-0.16	-0.16	-0.16	-0.15	-0.17	-0.17	-0.17	-0.17
	(-0.44)	(-0.46)	(-0.51)	(-0.57)	(-0.57)	(-0.58)	(-0.57)	(-0.61)	(-0.61)	(-0.62)	(-0.60)
Cokurt	-0.11	-0.11	-0.11	-0.11	-0.13	-0.16	-0.16	-0.15	-0.14	-0.14	-0.14
	(-1.50)	(-1.48)	(-1.45)	(-1.51)	(-1.72)	(-2.06)	(-2.07)	(-2.01)	(-1.90)	(-1.88)	(-1.78)
CIQ(τ)	0.68	0.73	0.89	1.30	2.26	0.55	0.88	0.87	0.13	0.03	-0.05
	(2.19)	(2.20)	(2.39)	(2.72)	(2.74)	(0.21)	(0.39)	(1.51)	(0.33)	(0.07)	(-0.13)
β^{DR}	-0.12	-0.12	-0.12	-0.11	-0.12	-0.14	-0.14	-0.13	-0.13	-0.13	-0.12
	(-1.17)	(-1.17)	(-1.15)	(-1.11)	(-1.18)	(-1.40)	(-1.41)	(-1.27)	(-1.29)	(-1.28)	(-1.25)
CIQ(τ)	0.96	1.01	1.18	1.63	2.69	0.51	0.69	0.68	-0.15	-0.24	-0.28
	(2.76)	(2.60)	(2.88)	(3.19)	(3.15)	(0.20)	(0.29)	(1.10)	(-0.34)	(-0.56)	(-0.64)
HTCR	119.53	118.76	118.64	119.47	118.91	111.84	113.06	118.60	118.29	116.58	114.63
	(3.00)	(2.98)	(2.97)	(2.97)	(2.91)	(2.75)	(2.77)	(2.88)	(2.92)	(2.96)	(2.94)
CIQ(τ)	0.80	0.80	0.84	1.06	1.68	-0.75	-0.73	0.15	-0.27	-0.32	-0.38
	(2.41)	(2.43)	(2.32)	(2.22)	(2.00)	(-0.28)	(-0.28)	(0.21)	(-0.55)	(-0.70)	(-0.80)
β^{-}	0.15	0.14	0.13	0.12	0.12	0.11	0.12	0.10	0.11	0.12	0.14
	(0.69)	(0.64)	(0.62)	(0.60)	(0.55)	(0.51)	(0.55)	(0.47)	(0.49)	(0.54)	(0.63)
CIQ(τ)	0.86	0.89	1.05	1.44	2.34	0.33	0.78	0.66	-0.15	-0.22	-0.27
	(2.45)	(2.26)	(2.46)	(2.73)	(2.61)	(0.12)	(0.32)	(1.09)	(-0.33)	(-0.51)	(-0.63)
DOWN ASY	-0.46	-0.46	-0.55	-0.56	-0.54	-0.58	-0.50	-0.43	-0.40	-0.27	-0.20
	(-0.23)	(-0.23)	(-0.26)	(-0.27)	(-0.26)	(-0.27)	(-0.24)	(-0.21)	(-0.19)	(-0.13)	(-0.10)
CIQ(τ)	1.07	1.10	1.26	1.58	2.65	1.26	1.03	1.13	0.13	-0.01	-0.15
	(2.70)	(2.56)	(2.68)	(2.61)	(2.54)	(0.40)	(0.37)	(1.55)	(0.25)	(-0.03)	(-0.32)
MCRASH	2.31	2.27	2.26	2.19	2.17	2.20	2.22	2.12	2.00	1.99	2.01
	(2.59)	(2.57)	(2.55)	(2.48)	(2.43)	(2.41)	(2.46)	(2.25)	(2.07)	(2.05)	(2.08)
CIQ(τ)	0.92	0.96	1.09	1.45	2.19	1.01	1.58	0.83	-0.12	-0.24	-0.33
	(2.70)	(2.62)	(2.66)	(2.72)	(2.40)	(0.39)	(0.66)	(1.40)	(-0.31)	(-0.64)	(-0.89)
COS PRED	-2.19	-2.26	-2.29	-2.40	-2.51	-2.53	-2.57	-2.50	-2.37	-2.40	-2.42
	(-1.30)	(-1.34)	(-1.34)	(-1.40)	(-1.47)	(-1.49)	(-1.52)	(-1.46)	(-1.39)	(-1.43)	(-1.44)

risk (MCRASH) of Chabi-Yo et al. (2022).[18] MCRASH possess significant predictive power for the cross-section, which does not erase the effect of common idiosyncratic risk on the

[18]We employ the baseline seven-factor version of their measure.

expected returns. Especially strong is the relationship between MCRASH and expected returns when controlling for CIQ(τ) risk in the left part of the joint distribution.

To control for the expectations of the coskewness, we also include stock-level predicted systematic skewness (COS PRED) of Langlois (2020) in the regressions. We can see that neither this variable drive out the effect of CIQ(τ) factors.

We also investigate whether the pricing information of the ΔCIQ(τ) factors is not subsumed by stock characteristics based on accounting and trading information.[19] To that end, we provide the results of the multivariate cross-sectional regressions, in which we simultaneously control stock-level characteristics such as size, book-to-price, net payout yield, turnover, illiquidity, profit, and investment. We report the results in Table 9. We can see that the additional variables do not erase the pricing effect of the CIQ(τ) risks. The downside factors are significant determinants of the risk premium peaking at $\tau = 0.3$ with t-statistics of 2.47. On the other hand, exposure to the upside factors do not carry any significant pricing information.

Table 10 summarizes the results of controlling for the effect of past returns on the cross-section. Same as in the case of previous set of variables, we report estimation results from multivariate regression including variables maximum return, momentum, intermediate return, and lagged return. We observe that the additional variables slightly diminish the effect of the ΔCIQ(τ) factors for extreme left tail (τ between 0.1 and 0.2) but the effect for non-extreme downside risk remain strong. The effect of upside quantile factors remain insignificant even in this setting.

To summarize this subsection, we have shown that the CIQ(τ) results from the Fama-MacBeth regressions suggest that the exposure to the idiosyncratic downside common events is significantly priced in the cross-section of stock returns, and that none of the discussed risks drives out the significance of these results. On the other hand, the exposure to the idiosyncratic upside potential captured by the quantile factors for $\tau \geq 0.5$ do not possess significant pricing implications for the cross-section of stock returns. This asymmetry further favors the hypothesis that the common volatility is not the reason behind the significant pricing consequences of the downside quantile factors.

4.2 Portfolio Sorts

Next, we asses performance of the ΔCIQ(τ) factors in terms of investment opportunities. To this end, we look at the returns of the portfolios sorted on the CIQ(τ) betas. Every month, we estimate CIQ(τ) betas for all stocks that possess 48 return observations during

[19] We construct the variables in the same vein as in Langlois (2020).

Table 9: *Fama-MacBeth regressions using CIQ(τ) factors and stock characteristics.* The table contains estimated prices of risk and t-statistics from the Fama-MacBeth predictive regressions. Each segment contains prices of risk of ΔCIQ(τ) betas while controlling for various stock characteristics. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$. Note the coefficients are multiplied by 100 for clarity of presentation.

	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9
CIQ(τ)	0.68	0.72	0.83	1.14	1.91	0.63	0.11	0.67	0.03	-0.04	-0.08
	(2.20)	(2.13)	(2.28)	(2.47)	(2.42)	(0.25)	(0.05)	(1.19)	(0.07)	(-0.11)	(-0.20)
Size	-0.02	-0.02	-0.02	-0.02	-0.02	-0.02	-0.02	-0.02	-0.02	-0.02	-0.02
	(-1.74)	(-1.75)	(-1.74)	(-1.76)	(-1.79)	(-1.89)	(-1.89)	(-1.73)	(-1.73)	(-1.76)	(-1.84)
Book-to-price	0.11	0.11	0.11	0.11	0.11	0.12	0.12	0.12	0.12	0.12	0.12
	(1.76)	(1.75)	(1.71)	(1.71)	(1.79)	(1.90)	(1.92)	(1.94)	(1.94)	(1.93)	(1.90)
Net payout yield	0.95	0.88	0.91	0.93	1.05	1.23	1.26	1.27	1.26	1.17	1.03
	(1.19)	(1.12)	(1.13)	(1.07)	(1.11)	(1.21)	(1.24)	(1.26)	(1.35)	(1.33)	(1.33)
Turnover	-0.10	-0.10	-0.10	-0.10	-0.11	-0.11	-0.11	-0.11	-0.10	-0.10	-0.10
	(-2.05)	(-2.13)	(-2.16)	(-2.21)	(-2.23)	(-2.07)	(-2.08)	(-2.13)	(-2.09)	(-2.13)	(-2.10)
Illiquidity	1.86	1.86	1.86	1.85	1.85	1.95	1.95	1.97	1.97	1.96	2.00
	(1.08)	(1.09)	(1.09)	(1.08)	(1.10)	(1.18)	(1.17)	(1.14)	(1.13)	(1.13)	(1.13)
Profit	0.47	0.46	0.47	0.47	0.47	0.47	0.47	0.48	0.48	0.48	0.48
	(3.60)	(3.59)	(3.61)	(3.61)	(3.56)	(3.56)	(3.57)	(3.68)	(3.71)	(3.71)	(3.68)
Investment	-0.39	-0.39	-0.38	-0.38	-0.39	-0.39	-0.39	-0.39	-0.39	-0.39	-0.39
	(-7.13)	(-7.09)	(-7.07)	(-7.16)	(-7.08)	(-7.20)	(-7.18)	(-7.22)	(-7.24)	(-7.26)	(-7.28)

Table 10: *Fama-MacBeth regressions using ΔCIQ(τ) factors and momentum-type characteristics.* The table contains estimated prices of risk and t-statistics from the Fama-MacBeth predictive regressions. Each segment contains prices of risk of CIQ(τ) betas while controlling for various momentum-type characteristics. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$. Note the coefficients are multiplied by 100 for clarity of presentation.

	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9
CIQ(τ)	0.49	0.52	0.63	0.88	1.79	0.83	0.13	0.72	0.13	0.10	0.08
	(1.66)	(1.69)	(1.90)	(2.11)	(2.49)	(0.37)	(0.06)	(1.31)	(0.34)	(0.29)	(0.25)
Maximum return	-11.17	-11.13	-11.12	-11.12	-11.12	-11.17	-11.22	-11.40	-11.49	-11.41	-11.24
	(-3.16)	(-3.15)	(-3.14)	(-3.05)	(-3.00)	(-2.99)	(-2.99)	(-3.09)	(-3.17)	(-3.21)	(-3.22)
Momentum	0.59	0.59	0.59	0.60	0.58	0.57	0.57	0.58	0.58	0.58	0.58
	(3.68)	(3.65)	(3.68)	(3.69)	(3.63)	(3.61)	(3.61)	(3.59)	(3.63)	(3.58)	(3.57)
Intermediate return	0.06	0.05	0.04	0.04	0.06	0.08	0.08	0.07	0.07	0.07	0.07
	(0.32)	(0.31)	(0.26)	(0.26)	(0.35)	(0.46)	(0.45)	(0.40)	(0.38)	(0.41)	(0.42)
Lagged return	-3.72	-3.73	-3.76	-3.75	-3.69	-3.64	-3.62	-3.60	-3.63	-3.65	-3.70
	(-7.05)	(-7.09)	(-7.12)	(-7.06)	(-6.89)	(-6.79)	(-6.73)	(-6.73)	(-6.84)	(-6.91)	(-7.07)

the last 60 months using data up to time t. We sort the stocks into ten portfolios based on their betas for every τ separately. We then record the portfolios' performances at time $t+1$ using either an equal-weighted or value-weighted scheme. Then we move one month ahead, re-estimate all the betas, and create new portfolios. We expect that, for $\tau < 0.5$, there will be an increasing pattern of returns from the low exposure to the high exposure portfolios, and vice versa for $\tau > 0.5$. The results for sorts based on ten portfolios summarizes Table 11. We observe an increasing return pattern for the portfolios with τ up to 0.4 for both equal-weighted and value-weighted schemes. This pattern practically disappears when we look at the portfolios formed on higher τ CIQ(τ) betas. This observation is in agreement

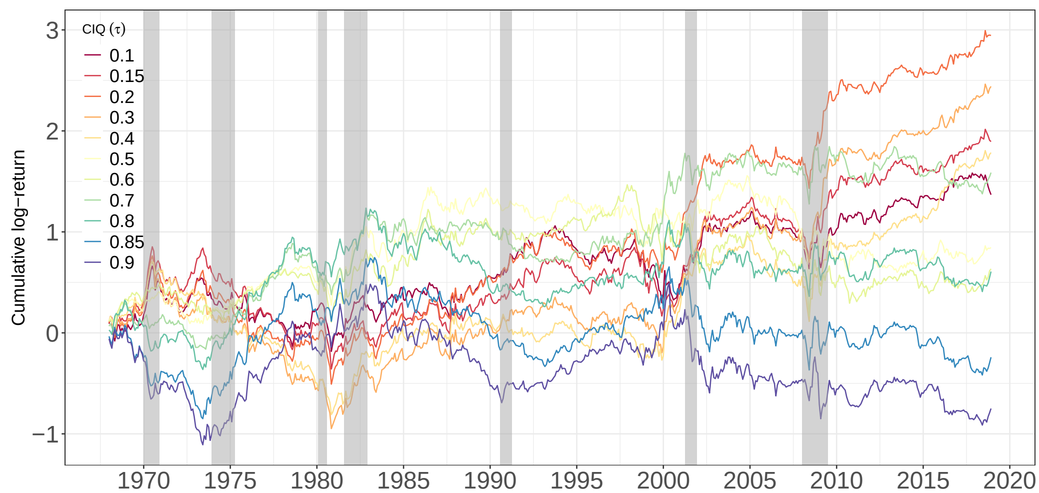
Table 11: *Portfolios sorted on the exposure to the $\Delta CIQ(\tau)$ factors.* The table contains annualized out-of-sample excess returns of ten portfolios sorted on the exposure to the ΔCIQ(τ) factors. We use all the CRSP stocks that have at least 48 monthly observations in each 60-month window. We report returns of the high minus low (H - L) portfolios, their t-statistics, and annualized 6-factor alphas with respect to the four factors of Carhart (1997), CIV shocks of Herskovic et al. (2016), and BAB factor of Frazzini and Pedersen (2014). We also report t-statistics for these alphas. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$.

τ	Low	2	3	4	5	6	7	8	9	High	H - L	t-stat	α	t-stat
							Equal-weighted							
0.10	4.88	7.89	8.20	9.19	8.75	9.47	9.99	10.65	10.71	9.63	4.74	2.76	4.80	2.55
0.15	4.41	7.78	8.88	8.95	9.59	9.39	10.11	10.36	10.10	9.79	5.38	3.07	5.60	3.04
0.20	4.37	7.43	8.82	9.33	9.10	10.14	10.11	10.06	9.86	10.12	5.74	3.18	6.36	3.28
0.30	4.41	7.54	8.49	9.15	9.87	9.71	10.15	10.28	10.16	9.59	5.19	3.10	5.76	3.37
0.40	4.65	8.11	8.82	9.39	9.05	9.64	10.22	10.35	9.90	9.22	4.57	2.95	4.79	3.01
0.50	6.77	9.01	9.95	9.38	9.29	9.48	9.61	9.73	9.11	7.02	0.25	0.14	-0.84	-0.41
0.60	6.35	9.25	9.77	9.16	9.66	9.75	9.84	9.20	8.96	7.42	1.07	0.63	-0.80	-0.43
0.70	6.28	8.84	9.86	9.11	9.19	9.01	9.54	9.40	9.57	8.55	2.27	1.61	0.15	0.09
0.80	8.05	9.34	9.43	9.02	8.84	9.39	9.23	8.91	8.78	8.36	0.32	0.20	-1.67	-0.96
0.85	8.19	9.13	9.54	8.97	9.02	9.40	9.61	8.88	8.57	8.03	-0.16	-0.10	-1.83	-0.99
0.90	8.14	9.69	9.40	8.89	9.11	9.58	9.32	8.89	8.87	7.45	-0.69	-0.38	-2.17	-1.13
							Value-weighted							
0.10	4.08	5.07	5.98	6.17	6.47	7.02	6.83	8.60	9.46	8.18	4.10	1.75	3.28	1.41
0.15	3.77	4.63	6.82	5.60	7.36	6.15	7.69	7.18	9.17	8.99	5.22	2.05	5.47	2.20
0.20	2.87	6.31	6.63	5.65	6.48	7.12	7.15	7.40	8.91	10.14	7.27	2.78	8.57	3.13
0.30	3.17	6.40	5.73	6.15	6.67	7.35	6.92	6.97	7.78	9.39	6.22	2.33	7.53	2.67
0.40	3.41	6.43	5.44	6.78	6.47	7.24	6.76	6.74	7.28	8.27	4.86	2.03	7.17	3.02
0.50	3.89	5.44	5.37	5.45	6.36	7.28	7.65	6.36	4.89	7.08	3.19	1.42	3.72	1.42
0.60	3.32	6.45	5.28	4.68	7.43	6.09	8.63	6.79	6.14	6.09	2.77	1.21	1.47	0.61
0.70	3.90	5.65	7.58	7.48	6.94	6.47	6.29	6.20	5.94	8.40	4.51	1.92	3.50	1.34
0.80	4.29	7.17	6.46	5.84	6.88	6.77	6.39	6.18	5.17	6.96	2.68	1.09	2.31	0.93
0.85	5.09	6.80	6.19	6.61	6.54	6.77	6.75	6.14	5.54	6.33	1.24	0.50	1.41	0.57
0.90	4.62	6.71	6.47	5.90	6.42	7.27	6.19	5.69	6.07	5.05	0.43	0.16	0.49	0.18

with the results from the Fama-MacBeth regressions and suggests that only the exposure to the lower tail common movements is priced in the cross-section.

Moreover, to formally assess whether there is a compensation for bearing a risk of high exposure to the common movements in various parts of distributions of idiosyncratic returns, we present returns of high minus low portfolios. We obtain these returns as a difference between returns of portfolios with the highest CIQ(τ) betas and portfolios with the lowest CIQ(τ) betas. These portfolios are zero-cost portfolios and capture the risk premium associated with specific τ joint movements of idiosyncratic returns. As expected, we observe a significant positive premium for the difference portfolios only for τ being less or equal to 0.4. These premiums are both economically and statistically significant. In the case of the equal-weighted portfolios, the premium for CIQ(0.2) factors is 5.74% on the annual basis with a t-statistic of 3.18. The premiums are very similar in the case of the value-weighted portfolios – e.g., for τ = 0.2 the premium is 7.27 with t-statistic of 2.78. This slightly lower significance in the case of the value-weighted portfolios may be partially caused by the

Figure 2: *Performance of the CIQ(τ) portfolios.* The figure depicts cumulative log-return of high minus low portfolios obtained from sorting the stocks into decile portfolios based on their exposure to the CIQ(τ) factors and buying the portfolio with high exposure and selling the portfolio with low exposure. Returns of the portfolios are value-weighted.



fact that the value-weighted portfolios possess a higher concentration, which leads to more volatile returns.

To make sure that the estimated premiums cannot be explained by exposure to other risks previously proposed in the literature, we regress the returns of the high minus low portfolios on four factors of Carhart (1997) and CIV shocks of Herskovic et al. (2016) and BAB factor of Frazzini and Pedersen (2014) and report corresponding annualized 6-factor alphas. From the results, we can see that the proposed factors do not capture the positive premium associated with the zero-cost portfolios. For the equal-weighted portfolio with $\tau = 0.2$, the estimated annualized alpha is 6.36% with t-statistic of 3.28, for value-weighted portfolios it is 8.57% premium with t-statistics being equal to 3.13.

To visually inspect the performance of the value-weighted CIQ(τ) portfolios, we present in Figure 2 cumulative log-return of the value-weighted high minus low portfolios for every τ. Consistent with the numerical results, only the portfolios based on CIQ factors for $\tau \leq 0.4$ provide strong performance during the sample period.

Next, in Table 12, we look at the performance of the CIQ(τ) sorted portfolios captured by the following twelve-month value-weighted returns. Each month, we construct portfolios as in the previous case. Instead of saving the next one-month return of the sorted portfolios, we record a twelve-month return, which follows after the formation period. Due to the passive approach for the following 12-month period, we focus on the value-weighted performance of the portfolios. We observe returns consistent with the results obtained using one-month returns. The high minus low portfolios with $\tau = 0.2$ yield 6.62% ($t = 2.43$). The other

Table 12: *Portfolio results with 1-year holding period.* The table summarizes annualized out-of-sample returns of the CIQ(τ) portfolios which are held for one year after their formation. The returns are value-weighted.

τ	Low	2	3	4	5	6	7	8	9	High	H - L	t-stat	α	t-stat
0.10	2.83	4.37	6.01	6.25	5.91	5.77	6.74	9.18	9.42	9.17	6.35	2.64	5.49	1.82
0.15	2.78	4.72	6.09	5.81	6.54	5.92	7.87	8.70	9.10	8.95	6.17	2.05	5.49	1.43
0.20	2.60	5.23	6.04	7.08	6.29	5.56	7.58	8.73	8.88	9.22	6.62	2.43	6.44	2.29
0.30	2.93	5.00	5.90	6.95	6.48	6.22	7.10	7.18	8.73	8.30	5.37	1.87	7.35	2.43
0.40	3.34	4.53	6.37	6.43	6.03	6.69	6.43	7.86	7.46	6.33	2.99	0.87	5.31	1.60
0.50	4.82	4.61	5.06	5.46	6.18	8.64	7.42	6.47	5.58	6.72	1.90	0.78	0.75	0.25
0.60	4.68	5.12	5.22	5.96	5.85	7.05	8.00	6.65	6.26	6.11	1.43	0.73	-0.79	-0.29
0.70	2.88	6.04	5.82	6.56	7.25	6.80	6.95	6.48	6.56	6.48	3.60	2.31	3.67	1.16
0.80	4.02	6.53	5.12	6.71	7.15	6.93	7.26	6.59	5.89	4.22	0.20	0.09	-0.36	-0.09
0.85	4.27	5.78	5.46	6.63	7.58	7.08	6.99	6.91	5.65	4.92	0.65	0.24	-0.96	-0.24
0.90	5.06	6.13	5.10	6.33	6.99	7.12	6.60	6.74	5.78	4.63	-0.43	-0.18	-2.78	-0.74

risk factors cannot explain these premiums as the 6-factor alphas stay economically and statistically significant.

Due to the fact that only the exposures to the lower tail common movements are priced, the previous results suggest that the CIQ(τ) risks are not driven by the effect of the common volatility. If it were the case that the volatility is the main driver of the obtained results, we would observe that both exposures to the lower and upper parts of the joint movements are priced, which is not the case. But to explicitly control for the effect of the common idiosyncratic volatility, we perform dependent bivariate sorts by double sorting on betas for PCA-SQ factor and betas for the ΔCIQ(τ) factors. Every month, we first sort the stocks into ten portfolios based on their PCA-SQ betas. Then, within each of the PCA-SQ-sorted portfolios, we sort the stocks into ten portfolios based on their CIQ(τ) betas. Finally, for each CIQ(τ) portfolio, we collapse all the corresponding CIV portfolios into one CIQ(τ) portfolio. This procedure yields single-sorted portfolios which vary in their CIQ(τ) betas but possess approximately equal PCA-SQ betas. The obtained results summarizes Table 13. For the equal-weighted portfolios, we see that the risk premium captured by the returns of the high minus low portfolios for $\tau \leq 0.4$ remains significant with an annualized return of 4.48% ($t = 3.14$) for $\tau = 0.2$. In case of the value-weighted portfolios, the return remain close to the equal-weighted case with return of 4.51% for $\tau = 0.2$ ($t = 2.21$.). These observations suggest that the CIQ(τ) risk premium captures risk that is not explained to the common volatility as described by the PCA-SQ model.

The portfolio results show that holding risk associated with the common idiosyncratic downside risk is rewarded by a significant premium. On the other hand, exposure to the common idiosyncratic upside potential is not related to robust pricing consequences. In Appendix A in Tables 17, 19, and 18, we provide results of the same analysis using five portfolios instead of ten. The results are qualitatively very similar to the results from the

Table 13: *Dependent bivariate sorts on CIQ(τ) and PCA-SQ exposures.* The table contains annualized out-of-sample excess returns of ten portfolios sorted on the exposure to the ΔCIQ(τ) and PCA-SQ factor. Exposure to the PCA-SQ factor are approximately same across the portfolios. We use all the CRSP stocks that have at least 48 monthly observations in each 60-month window. We report returns of the high minus low (H - L) portfolios, their t-statistics, and annualized 6-factor alphas with respect to the four factors of Carhart (1997), CIV shocks of Herskovic et al. (2016), and BAB factor of Frazzini and Pedersen (2014). We also report t-statistics for these alphas. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$.

τ	Low	2	3	4	5	6	7	8	9	High	H - L	t-stat	α	t-stat
							Equal-weighted							
0.10	6.09	7.97	7.92	9.28	9.06	9.71	9.37	9.51	10.35	10.13	4.05	3.01	4.04	2.71
0.15	6.09	7.76	8.73	8.87	9.03	8.93	10.16	9.51	10.28	10.02	3.92	2.75	3.85	2.52
0.20	5.63	8.49	8.14	9.01	9.46	9.27	9.53	9.70	10.08	10.11	4.48	3.14	4.60	2.83
0.30	5.43	8.40	8.30	8.95	9.18	9.92	9.26	9.76	10.30	9.94	4.51	3.19	4.34	2.83
0.40	5.57	8.76	8.64	8.94	9.10	9.40	9.22	10.10	10.09	9.60	4.03	2.70	3.60	2.26
0.50	6.77	9.69	9.26	9.66	9.37	9.56	9.20	9.31	9.09	7.48	0.71	0.44	-0.10	-0.05
0.60	6.16	10.00	9.56	9.35	9.63	9.11	9.93	8.70	9.32	7.66	1.49	0.96	0.14	0.09
0.70	6.52	8.93	9.40	9.65	8.81	9.07	9.34	9.37	9.19	9.13	2.61	2.05	1.44	1.06
0.80	7.74	9.21	9.36	9.34	8.94	8.93	8.47	9.11	9.41	8.87	1.14	0.89	-0.20	-0.14
0.85	7.68	9.10	9.06	8.86	9.16	9.03	8.99	9.61	9.02	8.86	1.18	0.87	0.11	0.07
0.90	7.89	9.45	8.89	9.14	9.10	9.15	9.25	8.81	9.17	8.54	0.65	0.45	-0.67	-0.42
							Value-weighted							
0.10	5.29	5.82	5.55	6.07	5.99	5.53	6.95	8.12	8.23	9.80	4.52	2.19	4.13	2.05
0.15	4.67	6.12	6.37	5.26	5.85	6.43	7.30	7.18	8.11	9.39	4.72	2.25	4.68	2.34
0.20	5.07	7.21	4.98	6.92	5.44	6.08	6.68	7.12	8.12	9.58	4.51	2.21	4.86	2.24
0.30	5.02	7.02	5.90	6.50	5.69	6.38	6.73	5.82	8.31	9.39	4.37	2.07	4.59	2.07
0.40	4.30	7.02	5.77	7.04	5.49	6.64	6.44	6.80	8.11	8.12	3.82	1.79	4.36	2.14
0.50	5.68	4.80	5.62	6.10	6.48	5.73	7.78	6.83	7.39	5.35	-0.33	-0.15	-0.85	-0.36
0.60	4.58	5.94	5.69	5.02	6.87	5.48	7.78	7.26	7.93	6.09	1.51	0.72	-0.18	-0.09
0.70	6.06	6.11	6.97	6.68	5.82	7.08	5.77	5.94	7.03	7.59	1.52	0.72	1.24	0.59
0.80	4.64	7.20	6.13	5.54	6.63	7.16	5.28	5.47	6.90	7.14	2.50	1.21	1.56	0.71
0.85	4.62	6.71	6.46	6.13	6.58	5.48	6.22	6.78	6.82	6.36	1.74	0.89	0.56	0.27
0.90	3.65	8.45	6.12	5.74	5.33	6.92	6.01	7.41	5.90	7.13	3.48	1.59	1.64	0.74

above, confirming the robustness of our claim that the exposure to the common left tail events is priced in the cross-section of returns.

Finally, we also repeat the exercise on the simulated universe of stocks. We simulate stocks from location-scale model in order to see that risk factors will not be quantile dependent, and will all be coming from the volatility. At the same time the exercise will show that choice of small sample in the moving window does not bias the results. Detailed discussion in Appendix A.1 shows that the premium associated with exposures to the different quantile levels on simulated data are the same to the exposures on the PCA-SQ factors in magnitude. The risk premiums have identical significance, and is constant (with opposite sign for downside and upside) over the quantiles. Hence if the returns were generated from the location-scale model, then quantile risk would be equivalent across quantiles, and it would be captured by the volatility risk.

Table 14: *Portfolios sorted on relative CIQ(τ) betas.* The table contains annualized out-of-sample excess returns of ten portfolios sorted on relative CIQ(τ) betas. We report returns of the high minus low (H - L) portfolios, their t-statistics, and annualized 6-factor alphas with respect to the four factors of Carhart (1997), CIV shocks of Herskovic et al. (2016), and BAB factor of Frazzini and Pedersen (2014). We also report t-statistics for these alphas. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$.

τ	Low	2	3	4	5	6	7	8	9	High	H - L	t-stat	α	t-stat
							Equal-weighted							
0.10	5.00	7.68	8.41	8.97	8.83	9.55	9.80	10.73	10.56	9.81	4.81	2.71	4.66	2.37
0.15	4.32	7.89	8.76	9.13	9.55	9.19	10.33	10.08	10.05	10.04	5.73	3.16	5.75	2.93
0.20	4.37	7.54	8.73	9.19	8.93	9.85	10.63	10.09	9.71	10.31	5.94	3.22	6.45	3.17
0.30	4.51	7.01	8.43	9.31	10.06	9.66	10.19	10.11	10.00	10.08	5.57	3.17	5.96	3.28
0.40	4.57	8.03	8.52	9.22	9.75	9.85	9.33	9.96	10.43	9.69	5.11	3.22	5.14	3.20
0.50	6.77	9.01	9.95	9.38	9.29	9.48	9.61	9.73	9.11	7.02	0.25	0.14	-0.84	-0.41
0.60	7.52	8.96	8.37	9.53	9.18	10.06	8.89	9.76	9.45	7.64	0.13	0.09	-1.27	-0.84
0.70	6.55	9.44	9.47	8.58	9.33	8.87	9.52	9.19	9.76	8.63	2.09	1.56	0.18	0.12
0.80	8.25	9.09	9.40	8.99	8.68	9.41	9.39	8.89	9.09	8.15	-0.10	-0.07	-1.94	-1.17
0.85	8.30	9.37	9.20	9.32	9.10	8.81	9.46	9.30	8.65	7.82	-0.48	-0.29	-1.88	-1.03
0.90	8.38	9.19	9.54	9.13	9.18	9.53	9.10	9.07	9.00	7.23	-1.15	-0.63	-2.45	-1.31
							Value-weighted							
0.10	3.83	4.53	6.11	6.53	6.49	6.33	6.95	8.51	9.70	8.48	4.66	1.97	2.96	1.29
0.15	4.00	4.72	6.40	6.60	7.20	6.30	7.03	7.39	9.33	9.12	5.11	2.04	4.75	1.94
0.20	2.79	6.25	6.36	6.38	6.61	7.11	7.12	7.23	8.59	9.68	6.89	2.72	7.12	2.74
0.30	3.47	6.24	5.64	6.02	7.85	6.96	6.89	6.53	8.18	9.66	6.19	2.33	6.83	2.50
0.40	3.69	6.37	6.24	6.53	7.16	6.83	6.09	6.27	7.96	8.99	5.30	2.06	7.35	2.87
0.50	3.89	5.44	5.37	5.45	6.36	7.28	7.65	6.36	4.89	7.08	3.19	1.42	3.72	1.42
0.60	5.93	5.56	5.97	5.30	5.70	5.80	6.53	7.13	7.77	6.82	0.89	0.39	0.23	0.08
0.70	4.40	6.63	5.92	7.42	6.85	7.24	5.54	5.88	6.68	7.34	2.94	1.26	2.02	0.77
0.80	5.38	7.48	5.50	6.67	6.78	6.74	6.51	6.12	5.01	6.45	1.07	0.43	0.65	0.25
0.85	5.07	7.05	6.70	6.34	6.51	6.51	6.75	6.29	5.63	5.54	0.48	0.18	0.75	0.29
0.90	4.96	6.84	6.30	6.58	6.26	6.76	6.87	5.37	5.72	5.57	0.62	0.22	1.53	0.55

4.3 Beyond CIQ(τ) Betas

To specifically capture additional information beyond median dependence from the lower and upper parts of the distribution, respectively, we define *relative* CIQ betas as

$$\beta_i^{rel}(\tau) := \beta_i(\tau) - \beta_i(0.5).$$

The results of the portfolio sorts based on relative betas are summarized in Table 14. These results are in the spirit of the CIQ betas' results presented above. The high minus low portfolio sorted on $\beta^{rel}(0.2)$ yields annual 5.94% excess return ($t = 3.22$) with 6-factor $\alpha = 6.45$ ($t = 3.17$) for the equal-weighted portfolio. In case of the value-weighted portfolios, we obtain annual return of 6.89% ($t = 2.72$) and $\alpha = 7.12$ ($t = 2.74$).

Because there is a little theory on which τ to choose when investing based on the exposure to the CIQ(τ) factors, we aim to aggregate the information from downside and upside factors into compressed measures. To summarize the dependence in the whole lower or upper part of the factor structure, we define *downside* and *upside* CIQ betas as

$$\beta_i^{down} := \sum_{\tau \in \tau_{down}} F\big(\beta_i(\tau)\big)$$
$$\beta_i^{up} := \sum_{\tau \in \tau_{up}} F\big(\beta_i(\tau)\big)$$

where $F\big(\beta_i(\tau)\big) = \frac{Rank(\beta_i(\tau))}{N_t+1}$. We obtain the downside and upside CIQ betas as an average cross-sectional rank of the CIQ betas for downside and upside τs, respectively. Results of the portfolio sorts based on those betas are summarized in Table 15. We can see that the long-short portfolios sorted on downside CIQ betas provide significant excess returns of 5.19% ($t = 3.02$) and 6.44% ($t = 2.48$) annual returns using equal- and value-weighted schemes, respectively. On the other hand, an investment strategy based on upside beta does not yield significant abnormal returns using either weighting approach.

To summarize the relative betas through the whole downside or upside parts of the joint structure, we introduce *downside* and *upside relative* betas

$$\beta_i^{down,rel} := \sum_{\tau \in \tau_{down}} F\big(\beta_i^{rel}(\tau)\big),$$
$$\beta_i^{up,rel} := \sum_{\tau \in \tau_{up}} F\big(\beta_i^{rel}(\tau)\big),$$

which are obtained as a mean cross-sectional rank of the relative betas associated with the exposure to the downside or upside CIQ(τ) factors, respectively. The associated returns are also summarized in Table 15. Similarly as in the case of the relative betas, downside relative betas provide investment strategy with significant abnormal returns of 6.02% ($t = 3.25$) and 7.40% ($t = 2.90$) on an annual basis using equal- or value-weighted returns, respectively. The returns of the portfolios based on relative upside betas remain insignificant.

5 Conclusion

We investigate the pricing implications of the exposures to the common idiosyncratic quantile factors. These factors capture non-linear common movements in various parts of the distributions across a large panel of stocks. Similarly, as the quantile regression extends the classical linear regression, our quantile factor model of asset returns extends the approximate factor models used in empirical asset pricing literature. We show that the downside quantile factors can robustly predict the market return out-of-sample. We also provide evidence that the expected returns are associated with the exposures to the downside common movements

Table 15: *Ten univariate sorted portfolios on combination CIQ betas.* The table contains annualized out-of-sample excess returns of ten portfolios sorted on downside (upside) and relative downside (upside) CIQ betas. We use all the CRSP stocks that have at least 48 monthly observations in each 60-month window. We report returns of the high minus low (H - L) portfolios, their t-statistics, and annualized 6-factor alphas with respect to the four factors of Carhart (1997), CIV shocks of Herskovic et al. (2016), and BAB factor of Frazzini and Pedersen (2014). We also report t-statistics for these alphas. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$.

Weighting	Variable	Low	2	3	4	5	6	7	8	9	High	H - L	t-stat	α	t-stat
Equal	β^{down}	4.71	7.20	8.54	9.23	9.48	10.20	9.26	10.49	10.34	9.90	5.19	3.02	5.66	3.19
	β^{up}	7.73	9.45	9.54	8.73	9.23	9.08	9.43	9.30	8.56	8.30	0.57	0.36	-1.43	-0.79
	$\beta^{down,rel}$	4.33	7.68	8.47	8.97	9.71	9.78	10.10	9.90	10.07	10.34	6.02	3.25	6.43	3.26
	$\beta^{up,rel}$	8.58	9.04	8.89	8.87	9.26	9.05	9.07	9.03	9.24	8.31	-0.27	-0.18	-2.00	-1.23
Value	β^{down}	3.08	6.41	5.90	5.85	5.72	8.06	6.96	7.59	8.31	9.52	6.44	2.48	7.15	2.88
	β^{up}	4.72	6.57	5.02	6.59	7.11	7.21	6.53	5.57	5.50	7.51	2.79	1.17	2.38	0.96
	$\beta^{down,rel}$	2.97	6.35	5.79	6.47	6.85	6.73	7.53	6.61	8.38	10.37	7.40	2.90	7.52	2.91
	$\beta^{up,rel}$	5.60	7.08	6.71	5.39	7.38	6.51	6.58	6.10	5.36	6.64	1.04	0.43	0.32	0.13

in contrast to the upside movements. Importantly, the quantile dependent factors provides richer information to investors in comparison to other downside risk or volatility factors. We perform various robustness checks to show that these results are not attributable to other previously proposed risk factors. Most notably, we aim to prove that the common volatility does not drive the results.

Future research may focus on better interpretability of the quantile factor models using the characteristics-based quantile factor model proposed by Chen et al. (2023). This investigation may identify which stock characteristics are related to exposure to common extreme events. From a theoretical perspective, future endeavors could explore the link between theoretical quantile asset pricing models, such as the model of Ramos et al. (2020), and quantile factor models. Furthermore, an important direction may extend the arbitrage pricing theory into the quantile domain in the spirit of Renault et al. (2022).

References

Aït-Sahalia, Y., J. Jacod, and D. Xiu (2020, November). Inference on risk premia in continuous-time asset pricing models. Working Paper 28140, National Bureau of Economic Research.

Allen, L., T. G. Bali, and Y. Tang (2012, 09). Does Systemic Risk in the Financial Sector Predict Future Economic Downturns? *The Review of Financial Studies 25*(10), 3000–3036.

Amengual, D. and E. Sentana (2020). Is a normal copula the right copula? *Journal of Business & Economic Statistics 38*(2), 350–366.

Amihud, Y. (2002). Illiquidity and stock returns: cross-section and time-series effects. *Journal of Financial Markets 5*(1), 31–56.

Andersen, T. G., N. Fusari, and V. Todorov (2020). The pricing of tail risk and the equity premium: Evidence from international option markets. *Journal of Business & Economic Statistics 38*(3), 662–678.

Ando, T. and J. Bai (2020). Quantile co-movement in financial markets: A panel quantile model with unobserved heterogeneity. *Journal of the American Statistical Association 115*(529), 266–279.

Ang, A., J. Chen, and Y. Xing (2006, 03). Downside Risk. *The Review of Financial Studies 19*(4), 1191–1239.

Ang, A., R. J. Hodrick, Y. Xing, and X. Zhang (2006a). The cross-section of volatility and expected returns. *The Journal of Finance 61*(1), 259–299.

Ang, A., R. J. Hodrick, Y. Xing, and X. Zhang (2006b). The cross-section of volatility and expected returns. *The journal of finance 61*(1), 259–299.

Annaert, J., M. De Ceuster, and K. Verstegen (2013). Are extreme returns priced in the stock market? european evidence. *Journal of Banking & Finance 37*(9), 3401–3411.

Bai, J. (2003). Inferential theory for factor models of large dimensions. *Econometrica 71*(1), 135–171.

Bai, J. and S. Ng (2002). Determining the number of factors in approximate factor models. *Econometrica 70*(1), 191–221.

Bali, T. G., N. Cakici, and R. F. Whitelaw (2011). Maxing out: Stocks as lotteries and the cross-section of expected returns. *Journal of Financial Economics 99*(2), 427–446.

Bali, T. G., N. Cakici, and R. F. Whitelaw (2014, 09). Hybrid Tail Risk and Expected Stock Returns: When Does the Tail Wag the Dog? *The Review of Asset Pricing Studies 4*(2), 206–246.

Bali, T. G., R. F. Engle, and S. Murray (2016). *Empirical asset pricing: The cross section of stock returns*. John Wiley & Sons.

Barro, R. J. (2006, 08). Rare Disasters and Asset Markets in the Twentieth Century*. *The Quarterly Journal of Economics 121*(3), 823–866.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics 31*(3), 307–327.

Bollerslev, T., A. J. Patton, and R. Quaedvlieg (2021). Realized semibetas: Disentangling "good" and "bad" downside risks. *Journal of Financial Economics*.

Bollerslev, T., A. J. Patton, and R. Quaedvlieg (2022). Granular betas and risk premium functions. *Available at SSRN 4258489*.

Bollerslev, T., G. Tauchen, and H. Zhou (2009, 02). Expected Stock Returns and Variance Risk Premia. *The Review of Financial Studies 22*(11), 4463–4492.

Bollerslev, T. and V. Todorov (2011). Tails, fears, and risk premia. *The Journal of Finance 66*(6), 2165–2211.

Campbell, J. Y. and S. B. Thompson (2007, 11). Predicting Excess Stock Returns Out of Sample: Can Anything Beat the Historical Average? *The Review of Financial Studies 21*(4), 1509–1531.

Carhart, M. M. (1997). On persistence in mutual fund performance. *The Journal of Finance 52*(1), 57–82.

Chabi-Yo, F., M. Huggenberger, and F. Weigert (2022). Multivariate crash risk. *Journal of Financial Economics 145*(1), 129–153.

Chabi-Yo, F., S. Ruenzi, and F. Weigert (2018). Crash sensitivity and the cross section of expected stock returns. *Journal of Financial and Quantitative Analysis 53*(3), 1059–1100.

Chamberlain, G. and M. Rothschild (1983). Arbitrage, factor structure, and mean-variance analysis on large asset markets. *Econometrica 51*(5), 1281–1304.

Chen, L., J. J. Dolado, and J. Gonzalo (2021). Quantile factor models. *Econometrica 89*(2), 875–910.

Chen, L., J. J. Dolado, J. Gonzalo, and H. Pan (2023). Estimation of characteristics-based quantile factor models.

Chen, X., L. P. Hansen, and J. Scheinkman (2009). Nonlinear principal components and long-run implications of multivariate diffusions. *Ann. Statist. 37*(1), 4279–4312.

Chernozhukov, V., I. Fernández-Val, and A. Galichon (2010). Quantile and probability curves without crossing. *Econometrica 78*(3), 1093–1125.

Connor, G. and R. A. Korajczyk (1986). Performance measurement with the arbitrage pricing theory: A new framework for analysis. *Journal of Financial Economics 15*(3), 373–394.

Corsi, F. (2009, 02). A Simple Approximate Long-Memory Model of Realized Volatility. *Journal of Financial Econometrics 7*(2), 174–196.

de Castro, L. and A. F. Galvao (2019). Dynamic quantile models of rational behavior. *Econometrica 87*(6), 1893–1939.

Ding, Y., R. F. Engle, Y. Li, and X. Zheng (2022). Factor modeling for volatility. *Available* *at SSRN 4282265*.

Dong, X., Y. Li, D. E. Rapach, and G. Zhou (2022). Anomalies and the expected market return. *The Journal of Finance 77*(1), 639–681.

Fama, E. F. and K. R. French (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics 33*(1), 3–56.

Fama, E. F. and J. D. MacBeth (1973). Risk, return, and equilibrium: Empirical tests. *Journal of Political Economy 81*(3), 607–636.

Farago, A. and R. Tédongap (2018). Downside risks and the cross-section of asset returns. *Journal of Financial Economics 129*(1), 69–86.

Frazzini, A. and L. H. Pedersen (2014). Betting against beta. *Journal of Financial Economics 111*(1), 1–25.

Giglio, S., B. Kelly, and S. Pruitt (2016). Systemic risk and the macroeconomy: An empirical evaluation. *Journal of Financial Economics 119*(3), 457–471.

Giglio, S., B. Kelly, and D. Xiu (2022). Factor models, machine learning, and asset pricing. *Annual Review of Financial Economics 14*.

Giglio, S., D. Xiu, and D. Zhang (2021, July). Test assets and weak factors. Working Paper 29002, National Bureau of Economic Research.

Gorodnichenko, Y. and S. Ng (2017). Level and volatility factors in macroeconomic data. *Journal of Monetary Economics 91*, 52–68.

Harvey, C. R. and A. Siddique (2000). Conditional skewness in asset pricing tests. *The Journal of Finance 55*(3), 1263–1295.

Herskovic, B., B. Kelly, H. Lustig, and S. Van Nieuwerburgh (2016). The common factor in idiosyncratic volatility: Quantitative asset pricing implications. *Journal of Financial Economics 119*(2), 249–283.

Hoerl, A. E. and R. W. Kennard (1970). Ridge regression: Applications to nonorthogonal problems. *Technometrics 12*(1), 69–82.

Hou, K. and R. K. Loh (2016). Have we solved the idiosyncratic volatility puzzle? *Journal of Financial Economics 121*(1), 167–194.

Jiang, L., K. Wu, and G. Zhou (2018). Asymmetry in stock comovements: An entropy approach. *Journal of Financial and Quantitative Analysis 53*(4), 1479–1507.

Jondeau, E., Q. Zhang, and X. Zhu (2019). Average skewness matters. *Journal of Financial Economics 134*(1), 29–47.

Kelly, B. and H. Jiang (2014, 06). Tail Risk and Asset Prices. *The Review of Financial*

Studies 27(10), 2841–2871.

Kelly, B. T., S. Pruitt, and Y. Su (2019). Characteristics are covariances: A unified model of risk and return. *Journal of Financial Economics 134*(3), 501–524.

Kozak, S., S. Nagel, and S. Santosh (2018). Interpreting factor models. *The Journal of Finance 73*(3), 1183–1223.

Langlois, H. (2020). Measuring skewness premia. *Journal of Financial Economics 135*(2), 399–424.

Lettau, M. and M. Pelger (2020). Factors that fit the time series and cross-section of stock returns. *The Review of Financial Studies 33*(5), 2274–2325.

Lu, Z. and S. Murray (2019). Bear beta. *Journal of Financial Economics 131*(3), 736–760.

Ma, S., O. Linton, and J. Gao (2021). Estimation and inference in semiparametric quantile factor models. *Journal of Econometrics 222*(1, Part B), 295–323. Annals Issue:Financial Econometrics in the Age of the Digital Economy.

Oh, D. H. and A. J. Patton (2017). Modeling dependence in high dimensions with factor copulas. *Journal of Business & Economic Statistics 35*(1), 139–154.

Pelger, M. and R. Xiong (2022). State-varying factor models of large dimensions. *Journal of Business & Economic Statistics 40*(3), 1315–1333.

Ramos, S. B., A. Taamouti, M. H. Lopes Moreira Da Veiga, and C.-W. Wang (2020, May). Quantile Consumption-Capital Asset Pricing. DES - Working Papers. Statistics and Econometrics. WS 30332, Universidad Carlos III de Madrid. Departamento de EstadÃ-stica.

Renault, E., T. Van Der Heijden, and B. J. M. Werker (2022, 04). Arbitrage Pricing Theory for Idiosyncratic Variance Factors*. *Journal of Financial Econometrics 21*(5), 1403–1442.

Rietz, T. A. (1988). The equity risk premium a solution. *Journal of Monetary Economics 22*(1), 117–131.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory 13*(3), 341–360.

Routledge, B. R. and S. E. Zin (2010). Generalized disappointment aversion and asset prices. *The Journal of Finance 65*(4), 1303–1332.

Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society: Series B (Methodological) 58*(1), 267–288.

Van Oordt, M. R. and C. Zhou (2016). Systematic tail risk. *Journal of Financial and Quantitative Analysis 51*(2), 685–705.

Welch, I. and A. Goyal (2007, 03). A Comprehensive Look at The Empirical Performance of Equity Premium Prediction. *The Review of Financial Studies 21*(4), 1455–1508.

Zou, H. and T. Hastie (2005). Regularization and variable selection via the elastic net. *Journal of the Royal Statistical Society: Series B (Statistical Methodology) 67*(2), 301–320.

A Appendix A

Table 16: *Correlations between CIQ(τ) and other factors.* The table reports correlations between CIQ(τ) factors and factors related to the asymmetric and variance risk. Data contain the period between January 1963 and December 2018.

variable / τ	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9
Panel A: *Levels of factors*											
PCA-SQ	-0.68	-0.66	-0.59	-0.47	-0.23	0.06	0.12	0.44	0.61	0.65	0.70
CIV	-0.29	-0.26	-0.24	-0.16	-0.03	0.04	0.07	0.19	0.28	0.28	0.30
TR	0.07	0.07	0.06	0.02	-0.04	-0.02	-0.04	-0.19	-0.19	-0.19	-0.17
VRP	0.04	0.05	0.06	0.07	0.10	-0.08	-0.03	0.03	0.03	0.00	0.01
VIX	-0.17	-0.15	-0.11	-0.01	0.16	0.08	0.13	0.30	0.33	0.30	0.28
Panel B: *Differences of factors*											
PCA-SQ	-0.54	-0.50	-0.44	-0.32	-0.11	0.17	0.17	0.35	0.51	0.55	0.60
CIV	-0.20	-0.17	-0.17	-0.12	-0.06	0.06	0.07	0.11	0.15	0.15	0.13
TR	0.11	0.09	0.09	0.04	-0.03	-0.03	-0.03	-0.24	-0.26	-0.27	-0.25
VRP	0.14	0.12	0.10	0.07	0.02	-0.05	-0.03	-0.06	-0.07	-0.11	-0.10
VIX	0.20	0.23	0.23	0.22	0.22	0.07	0.10	0.10	0.05	0.01	-0.06

Table 17: *Portfolios sorted on the exposure to the $\Delta CIQ(\tau)$ factors.* The table contains annualized out-of-sample excess returns of five portfolios sorted on the exposure to the ΔCIQ(τ) factors. We use all the CRSP stocks that have at least 48 monthly observations in each 60-month window. We report returns of the high minus low (H - L) portfolios, their t-statistics, and annualized 6-factor alphas with respect to the four factors of Carhart (1997), CIV shocks of Herskovic et al. (2016), and BAB factor of Frazzini and Pedersen (2014). We also report t-statistics for these alphas. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$.

τ	Low	2	3	4	High	H - L	t-stat	α	t-stat
					Equal-weighted				
0.10	6.38	8.69	9.11	10.32	10.17	3.78	2.59	4.36	2.79
0.15	6.10	8.91	9.49	10.23	9.94	3.85	2.58	4.65	3.05
0.20	5.90	9.08	9.62	10.09	9.99	4.08	2.78	5.20	3.47
0.30	5.97	8.82	9.79	10.21	9.88	3.90	2.81	4.89	3.56
0.40	6.38	9.11	9.34	10.29	9.56	3.19	2.55	4.00	3.22
0.50	7.89	9.67	9.38	9.67	8.07	0.18	0.14	-0.04	-0.03
0.60	7.80	9.46	9.70	9.52	8.19	0.39	0.30	-0.55	-0.42
0.70	7.56	9.49	9.10	9.47	9.06	1.50	1.31	0.29	0.22
0.80	8.69	9.23	9.11	9.07	8.57	-0.12	-0.10	-1.37	-0.98
0.85	8.66	9.26	9.21	9.24	8.30	-0.36	-0.26	-1.59	-1.07
0.90	8.91	9.14	9.35	9.11	8.16	-0.75	-0.48	-1.83	-1.22
					Value-weighted				
0.10	4.74	6.10	6.63	7.58	9.16	4.42	2.20	4.26	2.24
0.15	4.36	6.13	6.71	7.38	9.08	4.72	2.39	5.40	2.90
0.20	4.98	6.07	6.75	7.15	9.07	4.09	2.09	5.39	3.05
0.30	5.06	5.87	7.00	6.82	8.03	2.97	1.57	4.43	2.59
0.40	5.14	5.96	6.83	6.71	7.57	2.42	1.36	4.40	2.53
0.50	4.67	5.44	6.80	7.07	5.43	0.77	0.46	0.90	0.45
0.60	5.24	4.85	6.51	7.64	6.11	0.87	0.50	-0.16	-0.09
0.70	4.96	7.37	6.52	6.08	6.75	1.79	1.06	1.48	0.84
0.80	6.11	6.13	6.69	6.28	5.99	-0.11	-0.06	-0.49	-0.28
0.85	6.12	6.31	6.58	6.50	5.92	-0.20	-0.11	-0.39	-0.22
0.90	5.92	6.18	6.91	5.90	5.86	-0.06	-0.03	-0.20	-0.11

Table 18: *Dependent bivariate sorts on CIQ(τ) and PCA-SQ exposures.* The table contains annualized out-of-sample excess returns of five portfolios sorted on the exposure to the ΔCIQ(τ) and PCA-SQ factor. Exposure to the PCA-SQ factor are approximately same across the portfolios. We use all the CRSP stocks that have at least 48 monthly observations in each 60-month window. We report returns of the high minus low (H - L) portfolios, their t-statistics, and annualized 6-factor alphas with respect to the four factors of Carhart (1997), CIV shocks of Herskovic et al. (2016), and BAB factor of Frazzini and Pedersen (2014). We also report t-statistics for these alphas. Data contain period between January 1963 and December 2018. In each window, we use all the CRSP stocks that have at least 48 monthly observations, and we exclude penny stocks with prices less than 1$.

τ	Low	2	3	4	High	H - L	t-stat	α	t-stat
				Equal-weighted					
0.10	6.91	8.58	9.39	9.77	10.02	3.11	2.78	3.22	2.57
0.15	7.03	8.64	9.39	9.68	9.93	2.90	2.57	2.97	2.44
0.20	6.84	8.65	9.61	9.46	10.12	3.28	2.91	3.63	2.91
0.30	6.84	8.55	9.81	9.43	10.04	3.20	2.86	3.42	2.84
0.40	7.03	8.98	9.19	9.85	9.63	2.60	2.29	2.69	2.22
0.50	8.07	9.61	9.35	9.37	8.27	0.20	0.16	-0.33	-0.25
0.60	8.07	9.59	9.35	9.30	8.36	0.30	0.24	-0.75	-0.60
0.70	7.59	9.68	8.90	9.31	9.19	1.59	1.59	0.93	0.86
0.80	8.28	9.58	8.81	8.95	9.06	0.78	0.80	-0.03	-0.03
0.85	8.36	9.05	8.97	9.63	8.67	0.30	0.31	-0.50	-0.44
0.90	8.49	9.10	9.27	9.23	8.59	0.11	0.09	-0.85	-0.70
				Value-weighted					
0.10	5.43	5.79	6.05	6.96	8.58	3.15	2.07	2.20	1.68
0.15	5.58	6.15	5.90	7.25	7.77	2.19	1.43	1.76	1.24
0.20	6.08	5.88	5.97	6.80	7.91	1.83	1.30	1.95	1.42
0.30	6.21	6.09	6.25	6.22	8.04	1.83	1.23	2.11	1.50
0.40	5.56	6.56	6.15	6.75	7.08	1.51	0.96	1.99	1.30
0.50	5.15	5.58	6.25	7.02	6.42	1.27	0.80	0.28	0.16
0.60	5.55	5.27	6.08	7.41	7.22	1.67	0.99	0.19	0.11
0.70	6.01	6.41	6.58	5.68	7.56	1.55	0.98	0.55	0.39
0.80	5.77	6.12	6.28	6.15	6.65	0.88	0.59	-0.56	-0.37
0.85	5.81	6.12	6.53	6.29	6.48	0.67	0.49	-0.59	-0.40
0.90	5.15	6.63	6.36	6.49	6.71	1.56	0.97	0.05	0.03

Table 19: *Portfolio results with 1-year holding period.* The table summarizes returns of the CIQ(τ) portfolios which are held for one year after their formation. The returns are value-weighted.

τ	Low	2	3	4	High	H - L	t-stat	α	t-stat
0.10	3.71	6.13	5.82	8.00	9.41	5.70	2.56	6.49	1.70
0.15	4.01	5.79	6.15	8.21	9.12	5.11	2.28	6.69	1.88
0.20	4.15	6.48	5.80	8.10	8.98	4.83	2.25	7.00	1.93
0.30	4.14	6.40	6.34	7.25	8.40	4.26	2.06	5.14	1.33
0.40	3.97	6.44	6.18	7.10	7.11	3.15	1.73	2.81	0.91
0.50	4.43	5.19	7.40	6.90	5.94	1.51	1.13	-1.32	-0.75
0.60	4.80	5.33	6.37	7.21	6.26	1.46	1.12	-2.38	-1.47
0.70	5.17	6.11	6.94	6.57	6.37	1.20	0.91	-0.81	-0.60
0.80	5.82	5.86	6.85	6.92	5.21	-0.61	-0.38	-2.45	-1.92
0.85	5.45	6.14	7.11	6.83	5.21	-0.25	-0.13	-2.49	-1.84
0.90	5.82	5.94	6.99	6.46	5.32	-0.50	-0.27	-2.36	-1.52

A.1 Simulation Study

We present a simulation exercise to illustrate how the CIQ(τ) premiums would look like if the driving force behind them were simply common volatility. We simulate the returns from the following model

$$r_{i,t} = \alpha_i + \beta_i r_{m,t} + \gamma_i(V_t - \bar{V}) - \gamma_i \lambda^V + e_{i,t} \tag{18}$$

where V_t is the common variance factor, and the variance of the idiosyncratic error follows the factor structure proposed by Ding et al. (2022)

$$\begin{aligned} &e_{i,t} = \sqrt{V_{i,t}} z_{i,t}, \\ &V_{i,t} = V_t \exp(\mu_i + \sigma_i u_{i,t}) = V_t \tilde{V}_{i,t}, \\ &z_{i,t},\, u_{i,t} \sim i.i.d.\, N(0,1). \end{aligned} \tag{19}$$

Time-series variation of the returns drive two common factors – market factor, $r_{m,t}$, and variance factor V_t. The expected return of a stock is then equal to

$$\mathbb{E}[r_i] = \alpha_i + \beta_i \mathbb{E}[r_m] + \gamma_i \lambda^V. \tag{20}$$

We assume that the market factor follows a simple GARCH(1,1) process of Bollerslev (1986), which we fit on the market return from the empirical analysis. We assume that the log of the variance factor follows a modified HAR model of Corsi (2009)

$$\begin{aligned} \log V_{t+1} &= \theta_0 + \theta_m x_t^m + \theta_y x_t^y + v_{t+1} \\ v_{t+1} &\sim i.i.d.\, N(0, \sigma_v^2) \end{aligned} \tag{21}$$

where x_t^m and x_t^y are the previous month's log-variance and average log-variance over the last 12-month period, respectively. The common variance process is approximated by the cross-sectional average of the squared residuals from the time series regression of stock returns on the market factor. We fit the model from equation 21 on this time series. When simulating this time series, we initialize the process by randomly selecting 12 consequent observations of the common variance process estimated from the data and using those observations for iterating forward.

We calibrate the simulation setting to match the CRSP data sample we employ in the empirical investigation. We estimate stock-level market beta, β_i, using time-series regression of stock return on the market return. Exposure to the common variance, γ_i, is estimated by regressing the stock return on the estimate of the common variance process. Price of risk associated with the variance exposure, λ^V is chosen to be equal to 3×10^{-3}.[20] We estimate stock-level parameters of the idiosyncratic error variance–μ_i, σ_i–as the sample mean and standard deviation of $\log \tilde{V}_{i,t}$. To approximate the $\tilde{V}_{i,t}$, we use squared residuals from the time-series regression of the stock return on the market return. Then, to simulate these parameters, we approximate their distribution by normal distribution, with the mean equal to the estimates' cross-sectional average and the variance equal to the cross-sectional variance of the estimates.

We simulate the panel of 2,500 stocks with 120 observations. We repeat the simulation 1,000 times. Each time, we simulate stock returns by randomly choosing parameters for the stock-level process from the normal distribution with mean and variance corresponding to their sample counterparts. We remove the common time variation in stock returns by first forming the common linear factor

$$f_t = \frac{1}{N} \sum_{i=1}^{N} r_{i,t}, \quad t = 1, \dots, T \tag{22}$$

and then regressing the returns on this factor

$$r_{i,t} = \alpha_i + \hat{\beta}_i f_t + \hat{e}_{i,t}, \tag{23}$$

which yields the residuals $\hat{e}_{i,t}$. Those residuals are then used to form the common volatility and quantile factors. We construct the volatility factor as the first principal component of those squared residuals. ΔCIQ(τ) factors are estimated as discussed in Section 2. Exposures to those factors are then estimated using univariate time-series regressions of stock returns on the increments of the volatility or quantile factors, respectively.

Similarly, as in the empirical investigation, we sort stocks into decile portfolios based on their estimated exposure to the factors to infer the associated risk premiums. We proxy the premiums by computing high minus low returns of the portfolios. Table 20 reports the average premiums for all the CIQ(τ) factors. We observe that the premium is positive for the downside values of τ, negative for the upside ones and insignificant for the median. The magnitude of the premiums is comparable across all τ and, on average, in absolute value equal to 9.44%. The premium associated with the exposure to the PCA-SQ factor is -6.09%. We also compute associated t-statistics as a ratio between average premium and its standard deviation across all the simulations. All the premiums except for the median value

[20] This value corresponds to approximately 6% annual high minus low premium obtained from ten portfolios portfolios sorted on the exposure to the common variance. The choice of this value is not essential for the results that we present here.

Table 20: *Simulated risk premiums.* The table contains risk average premiums computed from high minus low returns of decile portfolios sorted on exposure to the CIQ(τ) risks. We simulate the returns using common variance factor model proposed by Ding et al. (2022). We simulate panel of 2,500 stocks with 120 monthly observations. We perform the simulation 1,000 times. t-statistics are obtained by dividing the average premium by its standard deviation. We also report proportion of rejections of non-significance of CIQ(τ) betas from multivariate cross-sectional regressions of average returns on those betas and market betas.

τ	Premium	t-stat	Rejections
0.10	9.34	2.62	0.96
0.15	9.37	2.56	0.96
0.20	9.38	2.51	0.96
0.30	9.50	2.54	0.97
0.40	9.37	2.26	0.96
0.50	0.35	0.03	0.96
0.60	-9.61	-2.70	0.96
0.70	-9.60	-2.72	0.96
0.80	-9.52	-2.67	0.96
0.85	-9.42	-2.58	0.96
0.90	-9.35	-2.56	0.96

are significant, with values around 2.6 in absolute value. The t-value associated with the PCA-SQ factor is -2.33. Next, we present the proportion of rejections of non-significance of CIQ(τ) betas at a 5% significance level from multivariate cross-sectional regressions of average returns on those betas and market betas. We can see that the proportions are virtually identical for both upside and downside betas of around 96%. The ratio for the PCA-SQ betas is 90%.

As we can see from the results, if there was a common volatility element present in the return, which is compensated in the cross-section, the CIQ(τ) risk premium would be symmetrical around the median. Moreover, the exposure to the PCA-SQ factor would be priced in this case. Overall, the evidence from the simulation exercise suggests that the CIQ(τ) risk premiums we observe in the data are not attributable to the common volatility compensation.